MATH 321:201: Real Variables II (Term 2, 2010)

**Home work assignment #3**

Due date: Friday, Jan 29, 2010 **(hand-in in class)**

**Problem 1** : Does the Mean Value Theorem hold also for vector-valued functions? Write the corresponding statement and show whether the statement is true or not.

**Problem 2** : Do [Rudin, Ch. 7. Exercise #5].

**Problem 3** : Do [Rudin, Ch. 7. Exercise #6].

**Problem 4** : Let $f_n, g_n : [0,1] \to \mathbf{R}$ be two sequences of functions, and $f$, $g$ are two bounded functions on $[0,1]$.

(a) Suppose $f_n$, $g_n$ converge *uniformly* to $f$, $g$ on $[0,1]$, respectively. Is it true that the product $f_n g_n$ converges also uniformly to $fg$?

(b) Assume instead $f_n$, $g_n$ are Riemann integrable and converge in $L^1$ to $f, g$ on $[0,1]$, respectively. Is it true that the product $f_n g_n$ converges also to $fg$ in $L^1$?

**Problem 5 (This problem is for extra mark! It can be worth of one homework set depending on your answer. Please hand-in this in separate sheets, but do not exceed three pages in a reasonable font size.)**: This is an essay question. Consider the upper Riemann-Stieltjes integral

$$\overline{\int_a^b} f d\alpha = \inf_{P \text{ partition of [a,b]}} U(P, f, \alpha)$$

as in Rudin 6.2.

Discuss what can be said about this ”upper integral” in the following way:

(a) Why is this definition meaningful? Why do you think people came up with this definition?

(b) Discuss what kind of theorems one can derive for $\overline{\int_a^b} f d\alpha$. Here you do not need to prove rigorously what you are claiming but give some reasons why those are plausible.

**The following are suggested exercises. Please DO NOT hand-in, *but, it is important for you to do these suggested exercises!***

**Problem:** Do Rudin, Ch. 7, Exercises #1, #2, #3, #4, #8.

**Problem ($L^1$ convergence does not imply pointwise convergence)**: Let $f_n : [0,1] \to \mathbf{R}$ be defined as follows:

$$f_n(x) = \chi_{[j2^{-k},(j+1)2^{-k}]}, \quad 0 \le j \le 2^k - 1, \ \ n = j + \sum_{i=1}^{k-1} 2^i.$$

Here, given a set $E$, $\chi_E$ denotes characteristic function of $E$, namely,

$$\chi_E(x) = \begin{cases} 1 & x \in E\ , \\ 0 & \text{otherwise.} \end{cases}$$

(a) Draw the graph of $f_n$ for $n = 1, 2, 3, \cdots, 10$.

(b) Show that $f_n$ converges to $f \equiv 0$ in $L^1$.

(c) Show that $f_n$ does not converge pointwise to $f \equiv 0$.

(d) Find a subsequence $f_{n_k}$ that converge pointwise to $f \equiv 0$.

**Problem:** In the above Problem #4, consider instead $\max[f_n, g_n]$ and answer the similar questions as in (a) and (b).