You have **FIVE** problems to hand-in. Hand in written solutions for grading at the BEGINNING of the lecture on the due date. Illegible, disorganized or partial solutions will receive no credit.

***Staple your HW. You will get *FIVE* marks OFF if you do not staple your HW! Note that the instructor will NOT provide stapler.**

Note: throughout the assignment, the function $u(t)$ denotes **the unit step function**:

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Also, in the following problems, feel free to use properties of Fourier Transform / Fourier Inversion and standard examples, e.g. Fourier transforms of $e^{-at}u(t)$ (for $a > 0$) and $\mathrm{rect}(t)$.

1. **(Scaling, time-shift, duality, differentiation)**
   (a) Find Fourier transform of

   $$f(t) = \begin{cases} t+1, & -1 \leq t \leq -1/2; \\ -t, & -1/2 \leq t \leq 0; \\ 0, & \text{otherwise.} \end{cases}$$

   (**Hint:** This is similar to one of class examples about differentiation rule for Fourier transform.)

   (b) Find Fourier transform of

   $$f(t) = \begin{cases} t+2, & -2 \leq t \leq -1; \\ -t, & -1 \leq t \leq 0; \\ 0, & \text{otherwise.} \end{cases}$$

   (**Hint:** Use (a) and scaling property of Fourier transform.)

   (c) Let $f(t) = e^{-|t|}$.
      i. Find $\widehat{f}(\omega)$. (**Hint:** this is a class example. You can use the result for $e^{-t}u(t)$ and apply properties of Fourier transform: here time-reversal property is relevant.)
      ii. Use part (i) and the duality property to find the Fourier transform $\widehat{g}(\omega)$ of the function

      $$g(t) = \frac{1}{\pi}\frac{1}{1+t^2}$$

2. (**Differentiation in frequency**)
   (a) Prove the following:

   $$\text{if } g(t) = tf(t) \text{ then } \widehat{g}(\omega) = i\tfrac{d}{d\omega}\widehat{f}(\omega)$$

   (**Hint:** differentiate the definition (I mean, the integral) of $\widehat{f}(\omega)$ with respect to ω: i.e.

   $$\frac{d}{d\omega}\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\frac{d}{d\omega}e^{-it\omega}dt. \quad)$$

   (b) Use (a) to show

   $$\text{if } \widehat{g}(\omega) = \tfrac{d}{d\omega}\widehat{f}(\omega), \text{ then } g(t) = -itf(t)$$

   (Here, use the fact that if $\widehat{f_1}(\omega) = \widehat{f_2}(\omega)$ then $f_1(t) = f_2(t)$. In other words, the Fourier transform $\widehat{f}(\omega)$ determine the original function $f(t)$.)

(c) Using the frequency differentiation property in part (a), compute the Fourier transform of:

(i) $f(t) = t\,\mathrm{rect}(t)$

(ii) $g(t) = t^2 e^{-3t} u(t)$ (**Hint:** you can apply the frequency differentiation property twice.)

(d) [Fourier inversion] For a real nonzero constant a, find the function $g(t)$ if

$$\widehat{g}(\omega) = \frac{1}{(i\omega + a)^2}$$

(**Hint:** You can use (b). Can you express $\widehat{g}(\omega)$ as a ω-derivative of certain function?)

3. **(RLC circuit)** Consider the ODE for RLC circuit:

$$LCy''(t) + RCy'(t) + y(t) = x(t)$$

(a) Let $R = 4$, $L = 3$, $C = 1$ and $\widehat{x}(\omega) = 1$. Find $y(t)$ using Fourier transform method.

(b) Let $R = 2$, $L = 1$, $C = 1$ and $\widehat{x}(\omega) = 1$. Find $y(t)$ using Fourier transform method. (**Hint:** You may want to use Problem 2 (d).)

(c) Let $R = 4$, $L = 3$, $C = 1$ and $x(t) = u(t)e^{-2t}$. (Note $u(t)$ is defined in the beginning of the HW). Find $y(t)$ using Fourier transform method.

4. **(Fourier Inversion)** In the following use properties of Fourier transform/inversion.

(a) $\widehat{f}(\omega) = \mathrm{sinc}(\omega + 1)$.

(b) Suppose that a function $\delta(t)$ satisfies $\widehat{\delta}(\omega) = 1$ and suppose $\frac{d}{dt}u(t) = \delta(t)$, where $u(t)$ is the unit step function defined in the beginning of the assignment. Express the inverse Fourier transform of

$$\widehat{h}(\omega) = \sin(\omega)$$

using $\delta(t)$ as well as its time-shift and scaling. (**Hint:** Find the relation to sinc function. Then, try to use properties of Fourier transform. You may have to express a rectangular function using the function $u(t)$. For example, $\mathrm{rect}(t) = u(t + 1/2) - u(t - 1/2)$. We will learn on Monday, what such function $\delta(t)$ is.)

(c) Use the same function $\delta(t)$ as in part (b) to express the inverse Fourier transform of

$$\widehat{g}(\omega) = \sin(5\omega + \pi/6).$$

(**Hint:** Use (b).)

5. **(Convolution)** *We will work out a couple of examples in class on Monday.*

Consider the functions

$$f(t) = \begin{cases} -2, & -2 \leq t < 0 \\ 1, & 0 \leq t < 4 \\ 0, & \text{otherwise} \end{cases}$$

$$g(t) = \begin{cases} 1, & -1 \leq t < 0 \\ 0, & \text{otherwise} \end{cases}$$

and $h(t) = (f * g)(\mathrm{t})$

(a) Find $h(t)$ and draw an accurate graph of this function on the interval $-4 \leq t \leq 5$. **Hint:** You should obtain a collection of straight line segments.

(b) Find $\widehat{h}(\omega)$. **Hint:** Use the convolution property, $\widehat{(f * g)}(\omega) = \widehat{f}(\omega)\widehat{g}(\omega)$.

(c) Compute the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)\mathrm{sinc}\left(\frac{\omega}{2}\right) e^{i\frac{3\omega}{2}}\, d\omega$$

(**Hint:** First, realize $\mathrm{sinc}(\omega/2)$ as Fourier transform of a function. Then, see what this integral means: it is the inverse Fourier transform of a certain function. At what t? Then, use a property of convolution.)

6. **(NOT TO HAND IN)**
Recall that the convolution of two functions f and g is the function $f * g$ defined by

$$(f * g)(t) = \int_{-\infty}^{\infty} f(s)g(t-s)\mathrm{d}s$$

and that $\widehat{(f * g)}(\omega) = \widehat{f}(\omega)\widehat{g}(\omega)$. Justify the following properties of the convolution:

(a) $f * g = g * f$

(b) For constants A_1 and A_2, we have $(A_1 f_1 + A_2 f_2) * g = A_1 (f_1 * g) + A_2 (f_2 * g)$

(c) $(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)$

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