Math 257 – Assignment 8. Due: Wednesday, March 16

Note that there are 6 problems. 2 pages.

1. **Heat conduction with derivative BC:** Consider the heat conduction problem:

$$u_t = \alpha^2 u_{xx}, \qquad 0 < x < L,\ t > 0;$$
$$\text{BC}: \quad u_x(0,t) = \alpha \text{ and } u_x(L,t) = \beta, \qquad t > 0;$$
$$\text{IC}: \quad u(x,0) = 0, \qquad 0 \le x \le L.$$

   a) Find the solution for $\alpha = \beta = 2$.

   b) For $\alpha = 1$, $\beta = 4$, find a particular solution $v(x,t)$ that satisfies the PDE and the boundary conditions, but not necessarily the initial conditions.

2. **Diffusion in a corridor:** There is a gas leak at one end of a long corridor $0 < x < 1$. The concentration of gas satisfies

$$u_t = u_{xx}, \qquad 0 < x < 1,\ t > 0$$

   with boundary conditions and initial condition

$$u_x(0,t) = 0, \quad u_x(1,t) = 1, \qquad t > 0;$$
$$u(x,0) = 0, \qquad 0 \le x \le 1.$$

   (The leak is at the end $x = 1$ and $u_x(1,t) = 1$ is the flux of gas being released). Find the solution for $u(x,t)$ by first searching for a particular solution of the form

$$w(x,t) = ax^2 + bx + ct$$

   that satisfies the equation and boundary conditions; then write $u(x,t) = w(x,t) + v(x,t)$, and find and solve the homogeneous problem for v.

   [*Hint:* $\int_0^1 x^2 \cos(n\pi x)dx = \frac{2}{n^2\pi^2}(-1)^n$ *for* $n \ge 1$]

   An alarm in the middle of the corridor ($x = 1/2$) is triggered when the gas concentration reaches 1. By considering the size of the different terms in your solution, deduce that the alarm goes off at approximately $t \approx 25/24$.

3. **Heat conduction with mixed boundary condition and heat source"** Solve the heat flow in a bar with an external source $Q(x) = x$:

$$u_t = u_{xx} + x, \qquad 0 < x < 1,\ t > 0;$$
$$u(0,t) = 0, \quad u_x(1,t) = 0, \qquad t > 0;$$
$$u(x,0) = -\frac{x^3}{6} + 1, \qquad 0 \le x \le 1.$$

   Hint: Again, look first for the steady-state solution of the problem.

4. **Finding the coefficients for a series expansion:** In March 9's class, we needed the following series expansion of

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n-1}{2L}\pi x\right) \tag{1}$$

where $f(x)$ is a piece-wise smooth function defined on the interval $0 \le x \le L$. In this problem, you are asked to verify that such series expansion of $f(x)$ exists and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2n-1}{2L}\pi x\right)dx.$$

(Note that this formula is specific to this series of $\sin\left(\frac{2n-1}{2L}\pi x\right)$*, and this is NOT a general formula for finding such coefficient for series expansion.)* Use the following steps.

   (a) Extend $f(x)$ to a function $g(x)$ defined on the interval $0 \le x \le 2L$, so that $g(L+x) = g(L-x)$ for $0 \le x \le L$ and $g(x) = f(x)$ for $0 \le x \le L$.

   (b) Make an odd extension $g_{odd}(x)$ of $g(x)$.

   (c) Compute the Fourier series of g_{odd} and justify (give a reason) that such Fourier series coincides with $f(x)$ on the interval $0 \le x \le L$, except at a finite number of points. Also, check that the Fourier series computed this way is exactly the same one given in (1).

5. **Heat conduction with mixed BC:** Find the solution to the following initial-boundary-value problem:

$$\begin{aligned} &u_t = 4\,u_{xx}, && 0 < x < 2,\ t > 0;\\ &BC: \quad u(0,t) = 1, \quad u_x(2,t) = 2, && t > 0;\\ &IC: \quad u(x,0) = 2x, && 0 \le x \le 2. \end{aligned}$$

6. **Eigenvalue problem:** Find all eigenvalues λ and eigenfunctions $y(x)$ of the eigenvalue problem:

$$\begin{aligned} -y''(x) &= \lambda\, y(x), && 0 < x < 2;\\ y'(0) &= 0,\\ y(2) &= 0. \end{aligned}$$