Chapter 10

Lecture 14 - Even and Odd Functions

Even: $f(-x) = f(x)$
Odd: $f(-x) = -f(x)$

10.1 Integrals of Even and Odd Functions

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx \tag{10.1}$$

$$= \int_{0}^{L} \big[f(-x) + f(x)\big]\,dx \tag{10.2}$$

$$= \begin{cases} 2\int_{0}^{L} f(x)\,dx & f \text{ even} \\ 0 & f \text{ odd.} \end{cases} \tag{10.3}$$

Notes: Let $E(x)$ represent an even function and $O(x)$ an odd function.

1. If $f(x) = E(x) \cdot O(x)$ then $f(-x) = E(-x)O(-x) = -E(x)O(x) = -f(x) \Rightarrow f$ is odd.

2. $E_1(x) \cdot E_2(x) \to$ even.

3. $O_1(x) \cdot O_2(x) \to$ even.

4. Any function can be expressed as a sum of an even part and an odd part:

$$f(x) = \frac{1}{2}\underbrace{\big[f(x)+f(-x)\big]}_{\text{even part}} + \frac{1}{2}\underbrace{\big[f(x)-f(-x)\big]}_{\text{odd part}}. \tag{10.4}$$

Check: Let $E(x) = \frac{1}{2}\big[f(x)+f(-x)\big]$. Then $E(-x) = \frac{1}{2}\big[f(-x)+f(x)\big] = E(x)$ even. Similarly let

$$O(x) = \frac{1}{2}\big[f(x)-f(-x)\big] \tag{10.5}$$

$$O(-x) = \frac{1}{2}\big[f(-x)-f(x)\big] = -O(x) \text{ odd.} \tag{10.6}$$

10.2 Consequences of Even/Odd Property for Fourier Series

(I) Let $f(x)$ be Even-Cosine Series:

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\Big(\frac{n\pi x}{L}\Big)}_{\text{even}}\, dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\Big(\frac{n\pi x}{L}\Big)\, dx \tag{10.7}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\Big(\frac{n\pi x}{L}\Big)}_{\text{odd}}\, dx = 0. \tag{10.8}$$

Therefore

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\Big(\frac{n\pi x}{L}\Big); \quad a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\Big(\frac{n\pi x}{L}\Big)\, dx \tag{10.9}$$

(II) Let $f(x)$ be Odd-Sine Series:

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\Big(\frac{n\pi x}{L}\Big)}_{\text{odd}}\,dx = 0 \qquad (10.10)\\ b_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\Big(\frac{n\pi x}{L}\Big)}_{\text{even}}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx \end{aligned}$$

Therefore

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{L}\Big); \quad b_n = \frac{2}{L}\int_{-0}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx.$$

(III) Since any function can be written as the sum of an even and odd part, we can interpret the cos and sin series as even/odd:

$$\begin{aligned} f(x) &= \underbrace{\frac{1}{2}\big[f(x)+f(-x)\big]}_{\text{even}} + \underbrace{\frac{1}{2}\big[f(x)-f(-x)\big]}_{\text{odd}} \qquad (10.11)\\ &= \left\{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\Big(\frac{n\pi x}{L}\Big)\right\} + \left\{\sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{L}\Big)\right\} \end{aligned}$$

where

$$\begin{aligned} a_n &= \frac{2}{L}\int_{0}^{L} \frac{1}{2}\big[f(x)+f(-x)\big]\cos\Big(\frac{n\pi x}{L}\Big)\,dx = \frac{1}{L}\int_{-L}^{L} f(x)\cos\Big(\frac{n\pi x}{L}\Big)\,dx\\ b_n &= \frac{2}{L}\int_{0}^{L} \frac{1}{2}\big[f(x)-f(-x)\big]\sin\Big(\frac{n\pi x}{L}\Big)\,dx = \frac{1}{L}\int_{-L}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,dx. \end{aligned}$$

10.3 Half-Range Expansions

If we are given a function $f(x)$ on an interval $[0, L]$ and we want to represent f by a Fourier Series we have two choices - a Cosine Series or a Sine Series.

Cosine Series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \tag{10.12}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx. \tag{10.13}$$

Sine Series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{10.14}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx. \tag{10.15}$$

Example 10.1 Expand $f(x) = x$ $0 < x < 2$ in a half-range (a) Sine Series, (b) Cosine Series.

(a)

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(t)\sin\frac{n\pi}{\ell}t\,dt \tag{10.16}$$

$$= \int_0^2 t\sin\frac{n\pi}{2}t\,dt \tag{10.17}$$

$$= -\left.\frac{t\cos\frac{n\pi}{2}t}{\left(\frac{n\pi}{2}\right)}\right|_0^2 + \frac{2}{n\pi}\int_0^2 \cos\frac{n\pi}{2}t\,dt \tag{10.18}$$

$$= -\frac{4}{n\pi}\cos(n\pi) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}t\right)\Big|_0^2 \tag{10.19}$$

$$= -\frac{4}{n\pi}(-1)^n \quad \begin{array}{rcl} f(1) & = & 1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{2}\right) \\ \frac{\pi}{4} & = & 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \end{array} \tag{10.20}$$

Therefore

$$f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{2}t\right). \tag{10.21}$$

(b)

$$a_0 = \frac{1}{2}\int_0^2 t\,dt = \frac{1}{2}\left.\frac{t^2}{2}\right|_0^2 = 1 \tag{10.22}$$

$$a_n = \int_0^2 t\cos\frac{n\pi}{2}t\,dt = \left(\frac{2}{n\pi}\right)t\sin\frac{n\pi}{2}t\Big|_0^2 - \left(\frac{2}{n\pi}\right)\int_0^2 \sin\frac{n\pi}{2}t\,dt$$

$$= +\left(\frac{2}{n\pi}\right)^2 \cos\frac{n\pi}{2}t\Big|_0^2 = \frac{4}{n^2\pi^2}\left\{\cos n\pi - 1\right\} \tag{10.23}$$

Therefore

$$f(t) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\left[(-1)^n - 1\right]}{n^2}\cos\frac{n\pi}{2}t \tag{10.24}$$

$$= 1 - \frac{8}{\pi^2}\sum_{n=0}^{\infty}\cos\frac{(2n+1)}{2}\pi t/(2n+1)^2. \tag{10.25}$$

The cosine series converges faster than Sine Series.

$$f(2) = 2 = 1 + \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \quad \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$