Math 100. Quiz 3. 2017-10-20 (Friday Q3-F-s) Time 25min

Section ......... Instructor name .............................................

Your email .................................................................

• For each computation of limits in this test, if the limit does not exist, indicate whether it diverges to $-\infty$ or $+\infty$.

• Simplify all your answers as much as possible and express answers in terms of fractions or constants such as $\frac{1}{100}$, $\sqrt{e}$ or $\ln(4)$ rather than decimals.
1. Each part of this question is worth 1 mark, and the correct answer will get the full mark.

(a) (1pt) Compute \( f'(t) \) for \( f(t) = (e^{2t} + t)^2 \)

**Solution.** Applying the chain rule:

\[
    f'(t) = 2(e^{2t} + t)(2e^{2t} + 1) = 2(e^{2t} + t)(2e^{2t} + 1)
\]

(b) (1pt) If \( x^3y^2 + y = e^x \), compute \( \frac{dy}{dx} \) at \( (x, y) = (0, 1) \).

We compute \( \frac{dy}{dx} \) by implicit differentiation. Differentiate both sides of the equation:

\[
    \frac{dy}{dx}(x^3y^2 + y) = \frac{dy}{dx}e^x
\]

Then

\[
    3x^2y^2 + x^3(2y)\frac{dy}{dx} + \frac{dy}{dx} = e^x
\]

Solving for \( \frac{dy}{dx} \):

\[
    \frac{dy}{dx} = \frac{e^x - 3x^2y^2}{2x^3y + 1} \quad (1)
\]

Note that \( (x, y) = (0, 1) \) satisfies the equation \( x^3y^2 + y = e^x \). So we can evaluate the derivative (1) at this point:

\[
    \left. \frac{dy}{dx} \right|_{(x,y)=(0,1)} = e^0 = 1.
\]
2. Each part of this question is worth 2 marks. **You have to show all your work in order to get credit.**

(a) **(2pt)** Suppose \( f(x) \) is a differentiable function such that \( f(1) = 1 \) and \( f'(1) = 3 \). Compute \( g'(1) \) where

\[
g(x) = f((f(x))^3)
\]

Applying the chain rule:

\[
g'(x) = f'(f(x)^3)(3f(x)^2)(f'(x))
\]

Next we evaluate at \( x = 1 \):

\[
g'(1) = f'(f(1)^3)(3f(1)^2)(f'(1)) = f'(1)(3 \cdot 1)(f'(1))
\]

and hence \( g'(1) = (3)^3 = 27 \)

(b) **(2pt)** Find all possible values for the constant \( C \) such that the tangent line to \( y = \arcsin(Cx) \) at \( x = 1 \) is parallel to the line \( 2y - x = 7 \).

**Solution.** First we find the derivative \( \frac{dy}{dx} \) by taking the derivative of both sides of the equation:

\[
\frac{dy}{dx} = \frac{dy}{dx} \arcsin(Cx) = \frac{C}{\sqrt{1 - (Cx)^2}}
\]

Evaluating at \( x = 1 \):

\[
\left. \frac{dy}{dx} \right|_{x=1} = \frac{C}{\sqrt{1 - C^2}} \quad (2)
\]

This is the slope of the tangent line at \( x = 1 \). The slope of the line \( 2y - x = 7 \) is \( \frac{1}{2} \). Recall that two lines are parallel if they have the same slope, so we are looking for the solutions to the equation

\[
\frac{C}{\sqrt{1 - C^2}} = \frac{1}{2} \quad (3)
\]

Equation (3) implies \( \frac{C^2}{1-C^2} = \frac{1}{4} \) (it is not equivalent, and the previous posted solution is incorrect). Simplifying, this is \( C^2 = 1/5 \). Therefore \( C = \pm \sqrt{1/5} \). Between these two values the one that satisfies (3) is \( C = \sqrt{1/5} \).
3. This question is worth 4 marks. **You have to show all your work in order to get credit.**

Consider the following equation

\[
\frac{x}{y - 1} = x^{y+1}
\]

Compute \(\frac{dy}{dx}\) at the point \((x, y) = (1, 2)\).

**Solution.** In order to compute \(y'\) we will use logarithmic differentiation.

\[
\log \left( \frac{x}{y - 1} \right) = \log (x^{y+1})
\]

Simplifying

\[
\log(x) - \log(y - 1) = (y + 1) \log(x)
\]

Next we differentiate

\[
\frac{d}{dx} \left( \log(x) - \log(y - 1) \right) = \frac{d}{dx} (y + 1) \log(x)
\]

Then

\[
\frac{1}{x} - \frac{1}{y - 1} \frac{dy}{dx} = \frac{y + 1}{x} + \log(x) \frac{dy}{dx}
\]

Solving for \(\frac{dy}{dx}\):

\[
\frac{dy}{dx} = \left( \frac{1}{x} - \frac{y + 1}{x} \right) \Bigg/ \left( \log(x) + \frac{1}{y - 1} \right)
\]

Note that \((x, y) = (1, 2)\) satisfies \(\frac{x}{y - 1} = x^{y+1}\), so we evaluate the derivative at this point:

\[
\left. \frac{dy}{dx} \right|_{(x, y) = (1, 2)} = (1 - 3) \Bigg/ (\log(1) + 1) = \frac{1 - 3}{0 + 1} = -2
\]
• For each computation of limits in this test, if the limit does not exist, indicate whether it diverges to $-\infty$ or $+\infty$.

• Simplify all your answers as much as possible and express answers in terms of fractions or constants such as $\frac{1}{100}$, $\sqrt{e}$ or $\ln(4)$ rather than decimals.
1. Each part of this question is worth 1 mark, and the correct answer will get the full mark.

(a) (1pt) Compute $f'(x)$ for $f(x) = \sqrt{1 + \cos(2\pi x)}$

Solution. Applying the chain rule:

$$f'(x) = \frac{-2\pi \sin(2\pi x)}{2\sqrt{1 + \cos(2\pi x)}} = \frac{-\pi \sin(2\pi x)}{\sqrt{1 + \cos(2\pi x)}}$$

(b) (1pt) If $xy + y^2x + 1 = x^2$, compute $\frac{dy}{dx}$ at $(x, y) = (1, 0)$.

Solution. We compute $\frac{dy}{dx}$ by implicit differentiation. Differentiate both sides of the equation:

$$\frac{d}{dx}(xy + y^2x + 1) = \frac{d}{dx}x^2$$

Then

$$y + x \frac{dy}{dx} + y^2 + x(2y) \frac{dy}{dx} = 2x$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{2x - y^2 - y}{x + 2xy} \quad (1)$$

Note that $(x, y) = (1, 0)$ satisfies the equation $xy + y^2x + 1 = x^2$. So we can evaluate the derivative (1) at this point:

$$\frac{dy}{dx}\bigg|_{(x,y)=(1,0)} = 2.$$
2. Each part of this question is worth 2 marks. **You have to show all your work in order to get credit.**

(a) **(2pt)** Suppose \( f(x) \) is a differentiable function such that \( f(1) = 1 \) and \( f'(1) = 2 \). Compute \( g'(1) \) where \( g(x) = f(f(x^3)) \)

**Solution.** Applying the chain rule:

\[
g'(x) = f'(f(x^3))f'(x^3)(3x^2)
\]

Next we evaluate at \( x = 1 \):

\[
g'(1) = f'(f(1^3))f'(1^3)(3 \cdot 1^2) = f'(f(1))f'(1)(3)
\]

and hence \( g'(x) = f'(1)f'(1)(3) = 12 \)

(b) **(2pt)** Find all possible values for the constant \( C \) such that the tangent line to \( y = C \arctan(Cx) \) at \( x = 1 \) is parallel to the line \( 3y - x = 1 \).

**Solution.** First we find the derivative \( \frac{dy}{dx} \) by taking the derivative of both sides of the equation:

\[
\frac{dy}{dx} = \frac{dy}{dx}C \arctan(Cx) = C \frac{dy}{dx} \arctan(Cx) = \frac{C^2}{1 + (Cx)^2}
\]

Evaluating at \( x = 1 \):

\[
\frac{dy}{dx} \bigg|_{x=1} = \frac{C^2}{1 + C^2} \quad (2)
\]

This is the slope of the tangent line at \( x = 1 \). The slope of the line \( 3y - x = 1 \) is \( \frac{1}{3} \). Recall that two lines are parallel if they have the same slope, so we are looking for the solutions to the equation

\[
\frac{C^2}{1 + C^2} = \frac{1}{3} \quad (3)
\]

Equation (3) is equivalent to \( 3C^2 = 1 + C^2 \). Simplifying, this is \( C^2 = 1/2 \). Therefore \( C = \pm \sqrt{1/2} \).
3. This question is worth 4 marks. You have to show all your work in order to get credit.

Consider the following equation

\[ 4xy = (x^2 + 1)^{y+1} \]

Compute \( \frac{dy}{dx} \) at the point \( (x, y) = (1, 1) \).

**Solution.** In order to compute \( y' \) we will use logarithmic differentiation.

\[
\log(4xy) = \log((x^2 + 1)^{y+1})
\]

Simplifying

\[
\log(4) + \log(x) + \log(y) = (y + 1) \log(x^2 + 1)
\]

Next we differentiate

\[
\frac{d}{dx} (\log(4) + \log(x) + \log(y)) = \frac{d}{dx} (y + 1) \log(x^2 + 1)
\]

Then

\[
\frac{1}{x} + \frac{1}{y} \frac{dy}{dx} = (y + 1) \frac{2x}{x^2 + 1} + \log(x^2 + 1) \frac{dy}{dx}
\]

Solving for \( \frac{dy}{dx} \):

\[
\frac{dy}{dx} = \left( \frac{2x(y + 1)}{x^2 + 1} - \frac{1}{x} \right) / \left( \frac{1}{y} - \log(x^2 + 1) \right)
\]

Note that \( (x, y) = (1, 1) \) satisfies \( 4xy = (x^2 + 1)^{y+1} \), so we evaluate the derivative at this point:

\[
\frac{dy}{dx} \bigg|_{(x,y)=(1,1)} = \left( \frac{2 \cdot 2}{2} - 1 \right) / (1 - \log 2) = \frac{1}{1 - \log 2}
\]