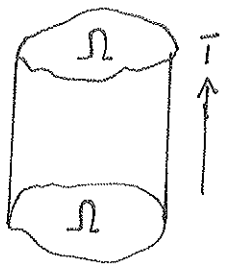
WAVE EQUATION

WE CONSIDER THE WAVE EQUATION WITH

$$u_{tt} - c^2 \Delta u = g(\xi, t) \quad \text{IN } \Omega$$

$$u = h \quad \text{ON } \partial\Omega$$

$$u = u_0(\xi) \text{ AND } u_t = u_1(\xi) \text{ AT } t = 0$$



WE WANT TO FIND A REPRESENTATION FOR THE SOLUTION IN TERMS OF A GREEN'S FUNCTION.

DEFINE $\quad L u \equiv c^2 \Delta u - u_{tt}. \qquad L^* u = c^2 \Delta u - u_{tt}$

THEN WE INTEGRATE OVER SPACE-TIME CYLINDER $C_T = \Omega \times [0, T]$

$$\int_{C_T} (c^2 \Delta u - u_{\tau\tau}) v \, d\xi \, d\tau = \int_0^T \left(\int_\Omega c^2 \Delta u \, v \, d\xi \right) d\tau - \int_\Omega \left(\int_0^T u_{\tau\tau} v \, d\tau \right) d\xi$$

NOW $\quad u_{\tau\tau} v = v_{\tau\tau} u + (u_\tau v - u v_\tau)_\tau$. THUS UPON USING THIS IDENTITY AND GREEN'S SECOND IDENTITY ON SPATIAL TERMS WE GET

$$\int_{C_T} (Lu) v \, d\xi \, d\tau = \int_0^T \left(\int_\Omega c^2 u \Delta v \, d\xi - \int_{\partial\Omega} c^2 (u \partial_n v - v \partial_n u) \, dS \right) d\tau$$

$$- \int_{C_T} u v_{\tau\tau} \, d\xi \, d\tau - \int_\Omega \left(\int_0^T (u_\tau v - u v_\tau)_\tau \, d\tau \right) d\xi$$

THIS YIELDS THAT

$$(1) \quad \int_{C_T} v Lu \, d\xi \, d\tau = \int_{C_T} u Lv \, d\xi \, d\tau - \int_0^T \left(\int_{\partial\Omega} c^2 (u \partial_n v - v \partial_n u) \, dS \right) d\tau.$$

$$- \int_\Omega (u_\tau v \Big|_0^T - u v_\tau \Big|_0^T) \, d\xi.$$

NOW WE IMPOSE CAUSALITY THAT $v_\tau \equiv v \equiv 0$ FOR $\tau > t$ AND THAT

$$(2) \quad \begin{cases} Lv = \delta(\xi - x, \tau - t) \\ v = 0 \quad \text{ON } \partial\Omega \\ v = v_\tau = 0 \quad \text{FOR } \tau > t \end{cases}$$

THIS YIELDS THAT

$$-\int_{C_t} v\, g(\xi,\tau)\,d\xi\,d\tau = u(x,t) - \int_0^t \Big(\int_{\partial\Omega} c^2 u\, \partial_n v\, dS \Big) d\tau + \int_\Omega \big(u_\tau v\big|_{\tau=0} - u\, v_\tau\big|_{\tau=0} \big)\, d\xi$$

WE SOLVE FOR $u(x,t)$ TO OBTAIN

$$u(x,t) = -\int_{C_t} v\, g(\xi,\tau)\,d\xi\,d\tau + c^2 \int_0^t \Big(\int_{\partial\Omega} u\, \partial_n v\, dS \Big) d\tau + \int_\Omega \big(u\, v_\tau\big|_{\tau=0} - u_\tau\, v\big|_{\tau=0} \big) d\xi$$

BUT NOW $u = u_0$ AND $u_\tau = u_1$ AT $\tau = 0$. ALSO $u = h$ ON $\partial\Omega$.

WE OBTAIN

$$u(x,t) = -\int_{C_t} v\, g(\xi,\tau)\,d\xi\,d\tau + c^2 \int_0^t \Big(\int_{\partial\Omega} u\, \partial_n v\, dS \Big) d\tau + \int_\Omega \big(u_0 v_\tau - u_1 v \big)\big|_{\tau=0}\, d\xi .$$

NOW WE HAVE THAT v SATISFIES (2). WE PUT

$$\sigma = t - \tau \,, \qquad v(\xi,\tau) = -G[\xi,\sigma]$$

AND USING THAT $\delta(\xi - x,\ \tau - t) = \delta(\xi - x,\ -\sigma) = \delta(\xi - x,\ \sigma)$. THEN WE OBTAIN

$$(3)\quad \begin{cases} G_{\sigma\sigma} - c^2 \Delta G = \delta(\xi - x,\ \sigma) \\ G = 0 \ \text{ ON } \partial\Omega \\ G = G_\sigma = 0 \ \text{ FOR } \sigma < 0 \end{cases}$$

IN TERMS OF G AND USING $v_\tau = +G_\sigma$ WE HAVE

$$(4)\quad \begin{cases} u(x,t) = \displaystyle\int_{C_t} G[\xi,\ t-\tau]\, g(\xi,\tau)\, d\xi\, d\tau - c^2 \Big|_0^t \Big(\int_{\partial\Omega} h\, \partial_n G[\xi,\ t-\tau]\, dS \Big) d\tau \\ \qquad\qquad + \displaystyle\int_\Omega u_0\, G_\sigma(\xi;\ t)\, d\xi + \int_\Omega u_1\, G(\xi;\ t)\, d\xi . \end{cases}$$

NOW PROBLEM (3) WAS STUDIED PREVIOUSLY IN THE NOTES:

WE FOUND THE FOLLOWING:

$$(5)\quad \begin{cases} G[\xi;\sigma] = \dfrac{1}{2c}\, H\left(\sigma - \dfrac{1}{c}|\xi - x|\right) & \text{IN 1-D.} \\ G[\underline{\xi};\sigma] = \dfrac{1}{2\pi c}\, \dfrac{H(\sigma - |\underline{\xi}-\underline{x}|/c)}{\sqrt{c^2\sigma^2 - |\xi - x|^2}} & \text{IN 2-D} \\ G[\underline{\xi};\sigma] = \dfrac{1}{4\pi c^2 |\underline{\xi}-\underline{x}|}\, \delta\left(\sigma - |\underline{\xi}-\underline{x}|/c\right) & \text{IN 3-D.} \end{cases}$$

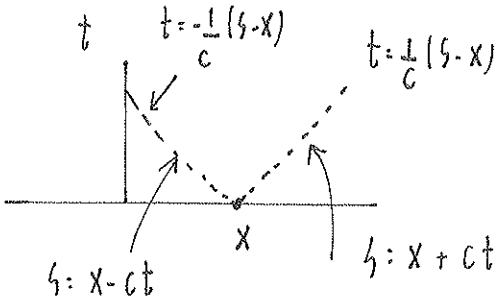
NOW CONSIDER 1-D

WE CONSIDER THE INFINITE DOMAIN PROBLEM SO THAT

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(\xi, t-\tau)\, g(\xi,\tau)\, d\xi\, d\tau + \frac{1}{2c}\int_{-\infty}^{\infty} u_0(\xi)\, \delta\left(t - \frac{1}{c}|\xi - x|\right) d\xi + \frac{1}{2c}\int_{-\infty}^{\infty} u_1\, H\left[t - \frac{1}{c}|\xi - x|\right] d\xi.$$

THIS GIVES

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{-\infty}^{\infty} g(\xi,\tau)\, H\left[(t-\tau) - \frac{1}{c}|\xi - x|\right] d\xi\, d\tau + \frac{1}{2c}\int_{-\infty}^{\infty} u_0(\xi)\, \delta\left(t - \frac{1}{c}|\xi - x|\right) d\xi + \frac{1}{2c}\int_{-\infty}^{\infty} u_1(\xi)\, H\left[t - \frac{1}{c}|\xi - x|\right] d\xi.$$



THUS

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} g(\xi,\tau)\, d\xi\, d\tau + \frac{1}{2c}\left[u_0(x+ct) + u_0(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(\xi)\, d\xi.$$

THIS IS THE SOLUTION TO $u_{tt} = c^2 u_{xx}$ IN $-\infty < x < \infty$

$u(x,0) = u_0(x)$, $u_t(x,0) = u_1(x)$.

NOW CONSIDER THE 3-D PROBLEM

WE WILL CONSIDER THE INFINITE DOMAIN PROBLEM AND FOR SIMPLICITY WE WILL SET $g \equiv 0$ TO CONSIDER THE "CAUCHY" PROBLEM

$$u_{tt} = c^2 \Delta u \quad \text{IN} \quad \mathbb{R}^3$$

WITH INITIAL DATA $u(\underline{x}, 0) = u_0(\underline{x})$ AND $u_t(\underline{x}, 0) = u_1(\underline{x})$.

THE SOLUTION FROM EQUATION (4) ON PAGE (W1) IS SIMPLY

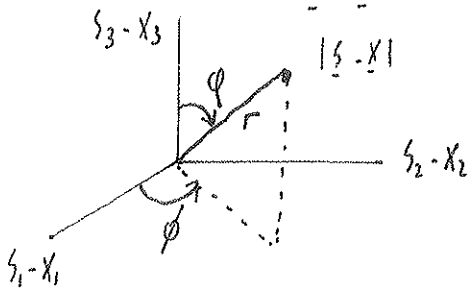
$$(*) \qquad u(\underline{x}, t) = \iiint_{\mathbb{R}^3} u_0(\underline{\xi})\, G_\sigma(\underline{\xi}, t)\, d\underline{\xi} + \iiint_{\mathbb{R}^3} u_1(\underline{\xi})\, G(\underline{\xi}; \underline{t})\, d\underline{\xi}$$

WHERE $$G(\underline{\xi}; \sigma) = \frac{1}{4\pi c^2 |\underline{\xi} - \underline{x}|}\, \delta\left(\sigma - \frac{|\underline{\xi} - \underline{x}|}{c}\right) = \frac{1}{4\pi c |\underline{\xi} - \underline{x}|}\, \delta(c\sigma - |\underline{\xi} - \underline{x}|)$$

UPON USING THE IDENTITY $\delta(z/b) = b\,\delta(z)$ WHEN $b > 0$.

WE NOW CALCULATE THE SECOND TERM IN $(*)$. WE INTRODUCE A SPHERICAL COORDINATE SYSTEM IN $\underline{\xi}$ SPACE CENTERED AT $\underline{x}$ SO THAT WITH $\underline{\xi} = (\xi_1, \xi_2, \xi_3)$ AND $\underline{x} = (x_1, x_2, x_3)$ WE HAVE WITH $r = |\underline{\xi} - \underline{x}|$

$$\xi_1 = x_1 + r \sin\varphi \cos\phi$$
$$\xi_2 = x_2 + r \sin\varphi \sin\phi$$
$$\xi_3 = x_3 + r \cos\varphi$$



THEN $d\underline{\xi} = r^2 \sin\varphi\, dr\, d\varphi\, d\phi$.

WE LET $d\Omega \equiv \sin\varphi\, d\varphi\, d\phi$. AND Ω BE THE SURFACE AREA OF UNIT SPHERE CENTERED AT $\underline{\xi} = \underline{x}$. THEN,

$$\iiint_{\mathbb{R}^3} u_1(\underline{\xi})\, G(\underline{\xi}; t)\, d\underline{\xi} = \int_\Omega \int_0^\infty u_1 \frac{1}{4\pi c r}\, \delta(ct - r)\, r^2\, dr\, d\Omega$$

$$= \frac{1}{4\pi c} \int_\Omega \left(\int_0^\infty u_1\, r\, \delta(ct - r)\, dr \right) d\Omega$$

$$= \frac{1}{4\pi c} \int_\Omega ct\, u_1\Big|_{|\underline{\xi} - \underline{x}| = ct} d\Omega = \frac{t}{4\pi} \int_\Omega u_1\Big|_{|\underline{\xi} - \underline{x}| = ct} d\Omega.$$

THUS WE HAVE

$$(\dagger)\qquad \iiint_{\mathbb{R}^3} u_1(\underline{\xi})\, G(\underline{\xi};t)\, d\underline{\xi} = \frac{t}{4\pi}\int_{\Omega} u_1\Big|_{|\underline{\xi}-\underline{x}|=ct} d\Omega .$$

IF WE WRITE THE INTEGRAL OUT EXPLICITLY WE HAVE

$$u_1(\underline{\xi})\Big|_{|\underline{\xi}-\underline{x}|=ct} = u_1\left[x_1 + ct\sin\varphi\cos\phi,\ x_2 + ct\sin\varphi\sin\phi,\ x_3 + ct\cos\varphi\right]$$

AND

$$\int_{\Omega} u_1\Big|_{|\underline{\xi}-\underline{x}|=ct} d\underline{\xi} = \int_0^{2\pi}\int_0^{\pi} u_1\left[x_1 + ct\sin\varphi\cos\phi,\ x_2 + ct\sin\varphi\sin\phi,\ x_3 + ct\cos\varphi\right]\sin\varphi\, d\varphi d\phi$$

NOW WE CALCULATE THE FIRST TERM IN $(\star)$ WE WRITE

$$G_\sigma(\underline{\xi};\sigma)\Big|_{\sigma=t} = G_t(\underline{\xi};t) \quad \text{SO THAT}$$

$$\iiint_{\mathbb{R}^3} u_0(\underline{\xi})\, G_t(\underline{\xi};t)\, d\underline{\xi} = \frac{\partial}{\partial t}\iiint_{\mathbb{R}^3} u_0(\underline{\xi})\, G(\underline{\xi};t)\, d\underline{\xi}$$

THEN WE CAN PROCEED EXACTLY AS BEFORE TO OBTAIN

$$\iiint_{\mathbb{R}^3} u_0(\underline{\xi})\, G_t(\underline{\xi};t)\, d\underline{\xi} = \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_{\Omega} u_0\Big|_{|\underline{\xi}-\underline{x}|=ct} d\Omega\right)$$

THEREFORE FROM $(\star)$ WE HAVE THE SOLUTION TO CAUCHY'S PROBLEM

$$u(\underline{x},t) = \frac{t}{4\pi}\int_{\Omega} u_1\Big|_{|\underline{\xi}-\underline{x}|=ct} d\Omega + \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_{\Omega} u_0\Big|_{|\underline{\xi}-\underline{x}|=ct} d\Omega\right)$$

THIS IS KNOWN IN THE LITERATURE AS KIRCHOFF'S SOLUTION. IT SATISFIES

$$u_{tt} = c^2\Delta u \quad \text{IN} \quad \mathbb{R}^3$$

WITH $u(\underline{x},0) = u_0(\underline{x})$ AND $u_t(\underline{x},0) = u_1(\underline{x})$

NEXT, WE CONSIDER THE WAVE EQUATION WITH ONLY A SOURCE TERM MODELLED BY

$$\begin{cases} u_{tt} = c^2 \Delta u + g(x,t) & \text{IN } \mathbb{R}^3 \\ u(x,0) = u_t(x,0) = 0 \end{cases}$$

THEN FROM (4) ON PAGE W2 WE OBTAIN

$$(S1) \qquad u(x,t) = \int_0^t \int_{\mathbb{R}^3} g(\xi,\tau)\, G[\xi, t-\tau]\, d\xi\, d\tau$$

WHERE

$$G[\xi;\sigma] = \frac{1}{4\pi c^2 |\xi - x|} \delta\left(\sigma - \frac{|\xi - x|}{c}\right) = \frac{1}{4\pi c |\xi - x|} \delta(c\sigma - |\xi - x|).$$

NOW WE INTRODUCE A SPHERICAL COORDINATE SYSTEM AS IN W4 CENTERED AT $\xi = x$. THEN (S1) BECOMES

$$u(x,t) = \int_0^t \int_\Omega \left(\int_0^\infty \frac{g(\xi,\tau)}{4\pi c r}\, \delta(c(t-\tau) - r)\, r^2\, dr \right) d\Omega\, d\tau$$

WE WILL DO THE INTEGRATION OVER τ FIRST. WE REWRITE

$$u(x,t) = \int_\Omega \int_0^\infty \frac{1}{4\pi c^2 r} \int_0^t \delta\left(t - \tau - r/c\right) r^2\, g(\xi,\tau)\, dr\, d\Omega\, d\tau$$

WE WRITE

$$u(x,t) = \int_\Omega \int_0^\infty \frac{1}{4\pi c^2 r} \left(\int_0^t \delta(t - \tau - r/c)\, g(\xi,\tau)\, d\tau \right) r^2\, dr\, d\Omega$$

WE HAVE

$$\int_0^t \delta(t - \tau - r/c)\, g(\xi,\tau)\, d\tau = \begin{cases} g(\xi, t - r/c) & \text{IF } 0 < t - r/c \\ 0 & \text{IF } t - r/c < 0. \end{cases}$$

THUS

$$\int_0^t \delta(t - \tau - r/c)\, g(\xi,\tau)\, d\tau = H[t - r/c]\, g(\xi, t - r/c)$$

WHERE

$$H(z) = \begin{cases} 1 & \text{IF } z > 0 \\ 0 & \text{IF } z < 0 \end{cases}.$$

NOW SUBSTITUTING YIELDS

$$u(\underline{x},t) = \frac{1}{4\pi c^2} \int_{\Omega} \int_{0}^{ct} r\, g(\underline{\xi}, t - r/c)\, dr\, d\Omega$$

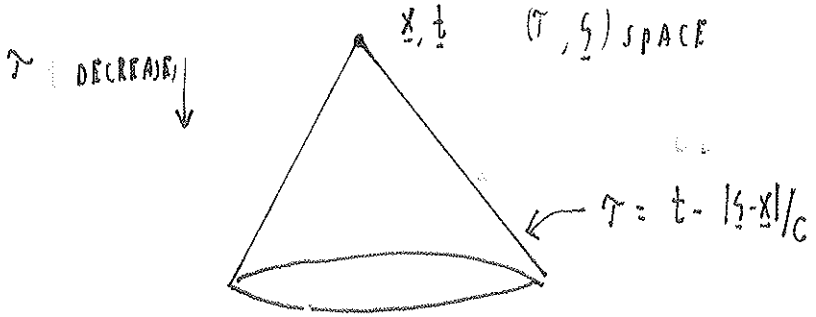
WE CAN WRITE THIS MORE COMPACTLY AS

$$u(\underline{x},t) = \frac{1}{4\pi c^2} \int_{V_t} \frac{g(\underline{\xi}, t - r/c)}{r}\, dV \qquad \text{WITH } r = |\underline{\xi} - \underline{x}|.$$

WHERE V_t IS THE SPHERE OF RADIUS ct IN THE $\underline{\xi}$-plane CENTERED AT POINT $\underline{x}$. THIS IS CALLED THE "LORENZ" SOLUTION.

THE SOLUTION IS CALLED THE RETARDED POTENTIAL.

THIS SOLUTION SHOWS THAT u AT POINT $\underline{x}$ AND TIME t DEPENDS ON VALUES OF g ON THE RETROGADE LIGHT CONE WITH VERTEX AT $\underline{x}$ AND t EACH CONTRIBUTION WAS EMMITED AT A TIME r/c EARLIER.



WAVE EQUATION IN 2-D

WE CONSIDER, $u_{tt} = c^2 \Delta u$ IN $\mathbb{R}^2$

$u(\underline{x}, 0) = u_0(\underline{x})$ AND $u_t(\underline{x}, 0) = u_1(\underline{x})$.

WE HAVE FROM (4) AND (5) ON PAGE (W2) AND (W3) THAT

$$u(\underline{x}, t) = \frac{\partial}{\partial t} \iint_{\mathbb{R}^2} u_0 \, G[\underline{\xi}; t] \, d\underline{\xi} + \iint_{\mathbb{R}^2} u_1 \, G[\underline{\xi}; t] \, dt$$

WHERE $$G[\underline{\xi}; t] = \frac{1}{2\pi c} \frac{H(t - |\underline{\xi} - \underline{x}|/c)}{\sqrt{c^2 t^2 - |\underline{\xi} - \underline{x}|^2}}$$

THUS WE HAVE

$$u(\underline{x}, t) = \frac{1}{2\pi c} \iint_{B_{ct}} \frac{u_1(\underline{\xi})}{\sqrt{c^2 t^2 - |\underline{\xi} - \underline{x}|^2}} \, d\underline{\xi} + \frac{1}{2\pi c} \frac{\partial}{\partial t} \iint_{B_{ct}} \frac{u_0(\underline{\xi})}{\sqrt{c^2 t^2 - |\underline{\xi} - \underline{x}|^2}} \, d\underline{\xi}$$

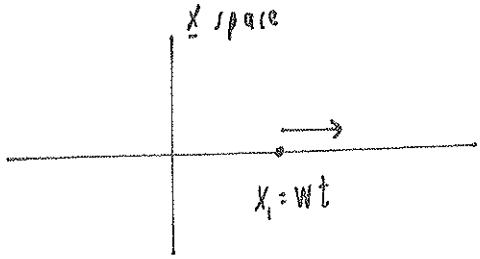
WHERE $$B_{ct} = \{ \underline{\xi} \mid |\underline{\xi} - \underline{x}| \leq ct \}.$$

THIS IS KNOWN AS "POISSON SOLUTION".

MOVING SOURCE PROBLEM

SUPPOSE THAT $u_{tt} - c^2 \Delta u = \delta(x_1 - wt)\,\delta(x_2)\,\delta(x_3)$

WITH $w > 0$. THIS REPRESENTS A MOVING SOURCE TRAVELLING IN x_1 DIRECTION AT SPEED w.



WE LOOK FOR A SOLUTION $\tilde{u}(y_1, y_2, y_3)$ WITH $y_1 = x_1 - wt$, $y_2 = x_2$, $y_3 = x_3$.

THEN $u_t = \tilde{u}_{y_1}(-w)$ $\quad u_{tt} = w^2 \tilde{u}_{y_1 y_1}$. THIS YIELDS THAT

$$w^2 \tilde{u}_{y_1 y_1} - c^2 \left[\tilde{u}_{y_1 y_1} + \tilde{u}_{y_2 y_2} + \tilde{u}_{y_3 y_3} \right] = \delta(y_1)\,\delta(y_2)\,\delta(y_3)$$

DIVIDE BY $-c^2$ TO OBTAIN

$$\tilde{u}_{y_1 y_1}\left(1 - w^2/c^2\right) + \tilde{u}_{y_2 y_2} + \tilde{u}_{y_3 y_3} = -\frac{1}{c^2}\,\delta(y_1)\,\delta(y_2)\,\delta(y_3).$$

NOW CHANGE VARIABLES TO GET THE LAPLACIAN.

$$y_1 = B\,\tilde{y}_1, \quad y_2 = \tilde{y}_2, \quad y_3 = \tilde{y}_3.$$

THEN $\tilde{u}_{y_1} = \hat{u}_{\tilde{y}_1}\,\frac{1}{B}$. IF WE CHOOSE $B = \sqrt{1 - w^2/c^2}$ FOR $w/c < 1$

THEN $\hat{u}_{\tilde{y}_1 \tilde{y}_1} + \hat{u}_{\tilde{y}_2 \tilde{y}_2} + \hat{u}_{\tilde{y}_3 \tilde{y}_3} = -\frac{1}{c^2}\,\delta\left(B\,\tilde{y}_1\right)\delta(\tilde{y}_2)\,\delta(\tilde{y}_3) = -\frac{1}{Bc^2}\,\delta(\tilde{y}_1)\,\delta(\tilde{y}_2)\,\delta(\tilde{y}_3)$

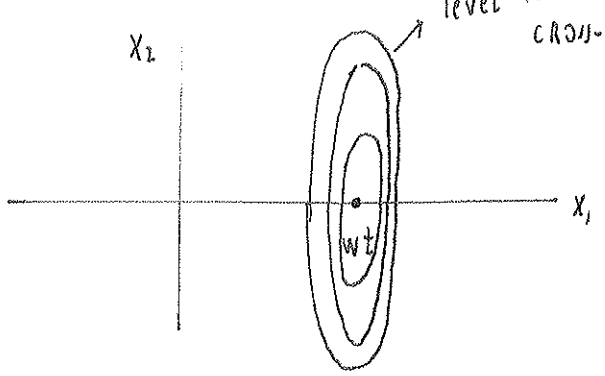
THUS $$\hat{u} = \frac{1}{4\pi B c^2}\,\frac{1}{\left[\tilde{y}_1^2 + \tilde{y}_2^2 + \tilde{y}_3^2\right]^{1/2}}$$

THUS $$\hat{u} = \frac{1}{4\pi c^2\left[B^2\tilde{y}_1^2 + \left(1 - w^2/c^2\right)\left(\tilde{y}_2^2 + \tilde{y}_3^2\right)\right]^{1/2}}$$

BUT $B^2\,\tilde{y}_1^2 = y_1^2 = (x_1 - wt)^2$ AND $\tilde{y}_2 = x_2$, $\tilde{y}_3 = y_3$.

THIS YIELDS THAT $$u(\underline{x}, t) = \frac{1}{4\pi c^2\left[(x_1 - wt)^2 + \left(1 - w^2/c^2\right)\left(x_2^2 + x_3^2\right)\right]^{1/2}}$$ FOR $0 < w < c$.

REMARK FOR THE CASE $0 < W < C$ WHERE THE SOURCE MOVES SLOWER THAN THE SPEED C ALLOWED BY THE MEDIUM WE HAVE THAT U IS CONSTANT ON THE ELLIPSOID

$$X_2^2 + X_3^2 + \frac{1}{1 - W^2/C^2}(X_1 - Wt)^2 = \text{CONSTANT} \qquad 0 < W < C.$$

WHICH IS CENTERED AT $(Wt, 0, 0)$.