POINT SOURCE SOLUTIONS OF WAVE EQUATION

WE WILL CALCULATE THE GREEN'S FUNCTION FOR THREE PROBLEMS

(1-D) (i) $$G_{xx} - \frac{1}{c^2} G_{tt} = \delta(x - x_0)\,\delta(t) \qquad -\infty < x < \infty,\ t \geq 0$$

$$G = G_t = 0 \quad \text{AT} \quad t = 0$$

(2-D) (ii) $$G_{rr} + \frac{1}{r} G_r - \frac{1}{c^2} G_{tt} = \frac{1}{2\pi r}\,\delta(r)\,\delta(t) \qquad 0 \leq r < \infty,\ t \geq 0$$

$$G = G_t = 0 \quad \text{AT} \quad t = 0$$

(3-D) (iii) $$G_{rr} + \frac{2}{r} G_r - \frac{1}{c^2} G_{tt} = \frac{1}{4\pi r^2}\,\delta(r)\,\delta(t) \qquad 0 \leq r < \infty,\ t > 0$$

$$G = G_t = 0 \quad \text{AT} \quad t = 0$$

THEY REPRESENT THE 1-D, 2-D AND 3-D CASES RESPECTIVELY. THE FORM OF THE GREEN'S FUNCTION WILL DEPEND ON THE DIMENSION.

IN (i), (ii) AND (iii) WE INTERPRET $\delta(t)$ AS $\delta(t-\varepsilon)$ FOR $\varepsilon \to 0^+$. THUS THE SOURCE IS TURNED ON AT $t = 0^+$ IN AN OTHERWISE QUIET MEDIUM.

TO SOLVE (i) TAKE LAPLACE TRANSFORMS $\hat{G}(x,s) = \mathcal{L}(G(x,t))$, $\mathcal{L}(\delta(t)) = 1$

$$\hat{G}_{xx} - \frac{s^2}{c^2}\hat{G} = \delta(x - x_0) \qquad -\infty < x < \infty$$

IMPOSING DECAY AS $|x| \to \infty$ GIVES

$$\hat{G}(x,s) = \begin{cases} A\, e^{-\frac{s}{c}(x - x_0)} & x > x_0 \\ A\, e^{\frac{s}{c}(x - x_0)} & x < x_0 \end{cases} \qquad (\text{RECALL } s > 0)$$

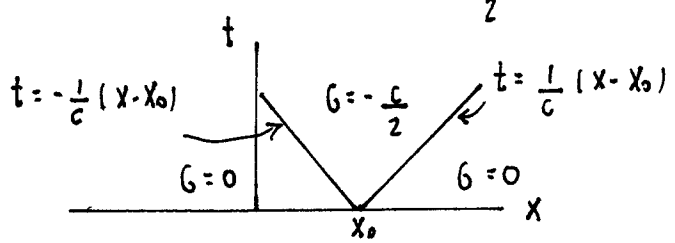
WE HAVE IMPOSED CONTINUITY AT $x = x_0$. THE JUMP CONDITION IS

$$\hat{G}_x(x_0^+) - \hat{G}_x(x_0^-) = 1 \quad \to \quad -2As/c = 1 \qquad A = -c/2s$$

THUS $$\hat{G}(x,s) = -\frac{c}{2s}\, e^{-s/c\,|x - x_0|}$$

NOW RECALL $\mathcal{L}^{-1}\left[e^{-rs}/s\right] = H(t - r)$, H IS HEAVISIDE FUNCTION.

THUS $$G(x,t) = -\frac{c}{2}\, H\left[t - \frac{1}{c}|x - x_0|\right]$$



TO SOLVE (ii) TAKE LAPLACE TRANSFORMS.

$$\hat{G}_{rr} + \frac{1}{r}\hat{G}_r - \frac{s^2}{c^2}\hat{G} = \frac{\delta(r)}{2\pi r}$$

THUS $\hat{G}(r,s) = A K_0(rs/c)$ WHERE $K_0(z)$ IS THE MODIFIED BESSEL FUNCTION SATISFYING $z^2 K_0'' + z K_0' - z^2 K_0 = 0$

$K_0(z) \sim -\log z$ AS $z \to 0$.

$\hat{G} \sim -A \log r$ AS $r \to 0$

$\partial\hat{G}/\partial r \sim -A/r$ AS $r \to 0$

TO FIND A, USE DIVERGENCE THEOREM

$$\int_0^{2\pi}\int_0^{\varepsilon} \Delta\hat{G}\, r\,dr\,d\varphi = \int_0^{2\pi} \left.\frac{\partial\hat{G}}{\partial r}\right|_{r=\varepsilon} \varepsilon\, d\varphi = 1 + \frac{s^2}{c^2}\int_0^{2\pi}\int_0^{\varepsilon}\hat{G}\, r\,dr\,d\varphi$$

LET $\varepsilon \to 0$, THEN

$$-A \cdot 2\pi = 1 \quad \to \quad A = -1/2\pi$$

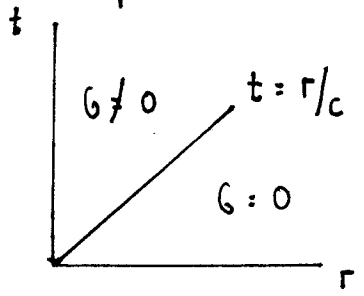
$$\hat{G}(r,s) = -\frac{1}{2\pi} K_0(rs/c)$$

NOW FROM A HANDBOOK

$$\mathcal{L}\left(\frac{1}{\sqrt{t^2-r^2}} H(t-r)\right) = K_0(rs)$$

THUS

$$G(r,t) = -\frac{c}{2\pi}\frac{H(t-r/c)}{\sqrt{c^2t^2-r^2}}$$



IF we draw a picture then

(i) NOTICE that ONCE the signal along $t = r/c$ HAS passed, the solution decays LIKE $\approx (c^2t^2-r^2)^{-1/2}$ FOR later times.

$\Rightarrow$ signal is NOT crisp and there is a "tail" that remains behind $\Rightarrow$ poor for sound communication

NOW SOLVE (iii). TAKE LAPLACE TRANSFORMS.

$$\hat{G}_{rr} + \frac{2}{r}\hat{G}_r - \frac{s^2}{c^2}\hat{G} = \frac{\delta(r)}{4\pi r^2}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\hat{G}_r\right) - \frac{s^2}{c^2}\hat{G} = \frac{\delta(r)}{4\pi r^2}$$

TO SOLVE THE HOMOGENEOUS PROBLEM LET $\hat{G} = f(r)/r$. THEN

$\frac{1}{r^2}\left(r^2\hat{G}_r\right)_r - \frac{s^2}{c^2}\hat{G} = 0$ REDUCES TO $f''(r) - \frac{s^2}{c^2}f(r) = 0$. IMPOSING

DECAY AS $r \to \infty$ GIVES $f(r) = A e^{-s/c\, r}$

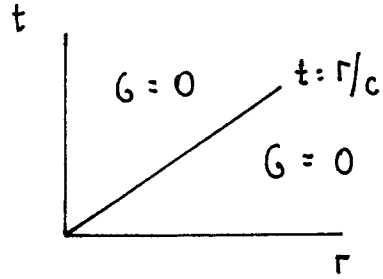
THUS $\hat{G}(r,s) = A/r\; e^{-sr/c}$

THE CONSTANT A CAN BE FOUND USING THE DIVERGENCE THEOREM TO GET

$$A = -1/4\pi$$

THUS $$\hat{G}(r,s) = -\frac{1}{4\pi r} e^{-sr/c}$$

INVERTING $$G(r,t) = -\frac{1}{4\pi r}\,\delta(t - r/c)$$



NOTICE THAT $G = 0$ BEHIND THE LINE $t = r/c \Longrightarrow$ SIGNAL IS CRISP!

HELMHOLTZ EQUATION

START WITH THE WAVE EQUATION

$$U_{tt} = c^2 \Delta U \qquad U = U(\underline{x}, t)$$

ASSUME THAT THERE IS A TIME PERIODIC DISTURBANCE OF THE FORM

$$U = e^{-i\omega t} u \qquad u = u(\underline{x})$$

THEN SUBSTITUTING GIVES HELMHOLTZ EQUATION

$$(*) \qquad \Delta u + k^2 u = 0 \qquad k = \omega/c \quad \text{wavenumber.}$$

NOW $(*)$ GOVERNS WAVE PROPAGATION WHEREAS $\Delta u - k^2 u = 0$ GOVERNS EXPONENTIAL DECAY (NO PROPAGATION).

NOW CALCULATE THE FREE-SPACE GREEN'S FUNCTION FOR

$$\Delta u + k^2 u = \delta(x - \xi, y - \eta) \qquad (2\text{-DIMENSIONS})$$

$$u_{rr} + \frac{1}{r} u_r + k^2 u = \frac{\delta(r)}{2\pi r}$$

THE SOLUTION TO THE homogeneous problem IS

$$u = A J_0(kr) + B Y_0(kr)$$

$J_0(z), Y_0(z)$ SATISFY

$$z^2 y'' + z y' + z^2 y = 0$$

$J_0(z)$ IS regular as $z \to 0$

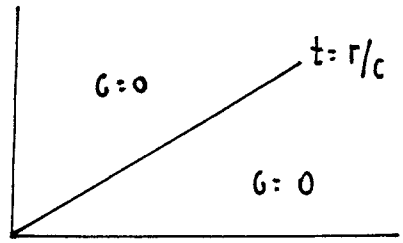
$Y_0(z) \sim +2/\pi \log z$ AS $z \to 0$.

THE CONSTANT A CAN BE FOUND USING THE DIVERGENCE THEOREM TO GET

$$A = -1/4\pi$$

THUS $$\hat{G}(r,s) = -\frac{1}{4\pi r} e^{-sr/c}$$

INVERTING $$G(r,t) = -\frac{1}{4\pi r}\delta(t - r/c)$$



NOTE: G = 0 BEHIND SIGNAL

$\Longrightarrow$ SIGNAL IS CRISP.

HELMHOLTZ EQUATION

START WITH THE WAVE EQUATION

$$(1) \qquad U_{tt} = c^2 \Delta U \qquad U = U(\underline{x}, t)$$

ASSUME THERE IS A PERIODIC DISTURBANCE OF THE FORM

$$U = e^{-i\omega t} u \qquad u = u(\underline{x})$$

SUBSTITUTING GIVES HELMHOLTZ' EQUATION

$$(2) \qquad \Delta u + k^2 u = 0 \qquad k = \omega/c \quad \text{wavenumber}$$

(2) GOVERNS WAVE propagation whereas $\Delta u - k^2 u = 0$ HAS exponential decaying AND GROWING SOLUTIONS. WE LOOK AT THIS PROBLEM IN (1-D) AND (2-D)

ONE-DIMENSION

SUPPOSE IN (1) WE HAD A LOCALIZED SOURCE, PERIODIC IN TIME

$$\Delta U - \frac{1}{c^2} U_{tt} = \delta(x - x_0) e^{-i\omega t}$$

THEN IF $U = e^{-i\omega t} u(x)$ WE GET

$$(3) \qquad u_{xx} + k^2 u = \delta(x - x_0)$$

THIS PROBLEM IS ILL-POSED IN THAT THERE IS <u>NO</u> UNIQUE SOLUTION. THIS CAN BE SEEN IN TWO WAYS

(i) SOLUTIONS OF $u_{xx} + k^2 u = 0 \rightarrow$ SINES AND COSINES both of which are BOUNDED AT INFINITY.

(ii) TAKING THE FOURIER TRANSFORM WE GET POLES ON THE AXIS OF INTEGRATION.

TO ELIMINATE THE NON-UNIQUENESS PROBLEM WE IMPOSE A PHYSICALLY BASED CONDITION THAT THE WAVES ARE RADIATING OUTWARD FROM THE SOURCE AS t INCREASES.

WITH A TIME FACTOR $e^{-i\omega t}$, THIS MEANS THAT

$$(4) \qquad u(x) \approx e^{i(k|x-x_0| - \omega t)} \quad \text{FOR} \quad |x| \to \infty \qquad \text{SOMMERFELD RADIATION CONDITION}$$

AND <u>NOT</u> $\quad u(x) \approx e^{-i(k|x-x_0| + \omega t)} \quad$ FOR $|x| \to \infty$.

NOW SOLVE (3) SUBJECT TO (4). IT IS CLEAR THAT WE WANT TO INTRODUCE COMPLEX-VALUED SOLUTIONS.

$$u(x) = \begin{cases} A e^{ik(x-x_0)} + B e^{-ik(x-x_0)} & x > x_0 \\ C e^{+ik(x-x_0)} + D e^{-ik(x-x_0)} & x < x_0 \end{cases}$$

FOR OUTGOINGNESS AS $x \to +\infty$ NEED $B = 0$

" " " $x \to -\infty$ NEED $C = 0$

CONTINUITY OF $u(x)$ AT $x = x_0 \longrightarrow A = D$.

$$(5) \qquad u(x) = \begin{cases} A e^{ik(x-x_0)} & x > x_0 \\ A e^{-ik(x-x_0)} & x < x_0 \end{cases}$$

NOW $\quad u'(x_0^+) - u'(x_0^-) = 1 \quad \longrightarrow \quad 2Aik = 1 \qquad A = -i/2k$

$$u(x) = -\frac{i}{2k} e^{ik|x-x_0|}$$

$$\text{OR} \qquad u(x) = \frac{1}{2k} e^{i(k|x-x_0| - \pi/2)}$$

NOW HOW CAN WE IMPOSE THE RADIATION CONDITION VIA A BOUNDARY CONDITION AS $|x| \to \infty$ RATHER THAN BY INSPECTION.

FROM (5) $\qquad u_x = iku \quad$ FOR $x > x_0$

$\qquad\qquad u_x = -iku \quad$ FOR $x < x_0$

THUS TAKE $\qquad u_x - iku = 0 \quad$ AS $x \to +\infty$

$\qquad\qquad u_x + iku = 0 \quad$ AS $x \to -\infty$

THESE RADIATION BOUNDARY CONDITIONS GUARANTEE OUTGOING WAVES AND THE SOLUTION TO (3) WITH (6) IS THEN UNIQUE.

<u>TWO-DIMENSIONS</u> $\Delta \Phi - \Phi_{tt}/c^2 = e^{-i\omega t}\,\delta(x-\xi, y-\eta)$

NOW LET $\Phi(\underline{x}, t) = e^{-i\omega t}\, u(\underline{x})$ AND CONVERT TO A POLAR COORDINATE SYSTEM CENTERED AT $x = \xi,\ y = \eta$. $r = [(x-\xi)^2 + (y-\eta)^2]^{1/2}$

$$u_{rr} + \frac{1}{r} u_r + k^2 u = \frac{\delta(r)}{2\pi r}$$

THERE ARE TWO SOLUTIONS TO THE PROBLEM (HOMOGENEOUS)

$$u = A J_0(kr) + B Y_0(kr)$$

<u>REMARK</u>

$$J_0(kr) \sim \left(\frac{2}{\pi k r}\right)^{1/2} \cos\left(kr - \pi/4\right) \quad \text{AS} \quad r \to \infty$$

$$Y_0(kr) \sim \left(\frac{2}{\pi k r}\right)^{1/2} \sin\left(kr - \pi/4\right) \quad \text{AS} \quad r \to \infty$$

$$Y_0(z) \sim \frac{2}{\pi} \log z \quad \text{AS} \quad z \to 0 \qquad J_0(z) \sim 1 \quad \text{AS} \quad z \to 0.$$

IF WE IMPOSE THE SINGULARITY CONDITION WE GET FROM THE DIVERGENCE THEOREM THAT

$$u = A J_0(kr) + \frac{1}{4} Y_0(kr)$$

A IS ARBITRARY (NON-UNIQUENESS)

THE FUNCTION u IS BOUNDED FOR ALL r.

TO ELIMINATE THE NON-UNIQUENESS WE IMPOSE AN OUTGOING RADIATION CONDITION WITH RESPECT TO TIME FACTOR $e^{-i\omega t}$. IT IS CONVENIENT TO INTRODUCE COMPLEX VALUED SOLUTIONS AS FOLLOWS:

DEFINE
$$H_0^{(1)}(z) = J_0(z) + i\, Y_0(z)$$
$$H_0^{(2)}(z) = J_0(z) - i\, Y_0(z)$$

THEN
$$H_0^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{i(z - \pi/4)} \quad \text{AS} \quad z \to +\infty \qquad H_0^{(1)}(z) \sim \frac{2i}{\pi} \log(z)$$
$$H_0^{(2)}(z) \sim \left(2/\pi z\right)^{1/2} e^{-i(z - \pi/4)} \quad \text{AS} \quad z \to \infty \qquad \text{AS} \quad z \to 0$$

THUS TO ENSURE THAT u IS OUTGOING WE HAVE

$$u = A H_0^{(1)}(kr) \quad \text{WITH} \quad u \sim \frac{2i}{\pi} A \log r \quad \text{AS } r \to 0$$

WE WANT $u \sim \frac{1}{2\pi} \log r$ AS $r \to 0$ THUS $A = -i/4$

THUS $u = -\frac{i}{4} H_0^{(1)}(kr)$ IS OUTGOING FREE SPACE GREEN'S FUNCTION

SINCE $u \sim -\frac{i}{4} \left(\frac{2}{\pi k r}\right)^{1/2} e^{i(kr - \pi/4)}$ AS $r \to \infty$

WE CAN IMPOSE AN OUTGOING CONDITION BY DEMANDING THAT

$$\partial_r u - ikU \to 0 \quad \text{AS} \quad r \to \infty$$

<u>SUMMARY</u>: THE PROBLEM $u_{rr} + \frac{1}{r} u_r + k^2 u = \frac{\delta(r)}{2\pi r}$

WITH $u_r - iku \to 0$ AS $r \to \infty \Longrightarrow$ SOMMERFELD RADIATION CONDITION.

HAS A UNIQUE SOLUTION $u = -\frac{i}{4} H_0^{(1)}(kr)$ WHICH IS OUTGOING AS $r \to \infty$.

THEN

$$U(r,t) = -\frac{i}{4} H_0^{(1)}(kr) e^{-i\omega t} \sim -\frac{i}{4} \left(\frac{2}{\pi k r}\right)^{1/2} e^{i(kr - \omega t - \pi/4)} \quad \text{AS } r \to \infty.$$

<u>REMARK</u>: IF WE IMPOSE $u_r + iku \to 0$ AS $r \to \infty$ WE WOULD GET A UNIQUE <u>INGOING</u> SOLUTION $u = -\frac{i}{4} H_0^{(2)}(kr)$

<u>THREE-DIMENSIONS</u> IN 3-D WE HAVE

$$U_{rr} + \frac{2}{r} U_r - U_{tt}/c^2 = e^{-i\omega t} \delta(r)/4\pi r^2$$

NOW LET $U[r,t] = u(r) e^{-i\omega t}$. THEN

$$(*) \quad Lu \equiv u_{rr} + \frac{2}{r} u_r + k^2 u = \delta(r)/4\pi r^2 \qquad 0 < r < \infty$$

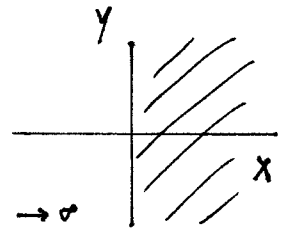
THERE ARE TWO SOLUTIONS TO $Lu = 0$ $u = e^{ikr}/r,\ e^{-ikr}/r$.

THE SOLUTION WHICH IS OUTGOING AT INFINITY IS

$u = A e^{ikr}/r$ SINCE $u \sim 1/4\pi r$ AS $r \to 0$ WE NEED $A = -1/4\pi$.

THE <u>OUTGOING</u> SOLUTION IS $u = -1/4\pi r \, e^{ikr}$. BY PUTTING IN AN RADIATION BOUNDARY CONDITION AS IN 2-D WE HAVE $u_r - iku \to 0$ AS $r \to \infty$

$\cancel{u_r - 1/r\, u}$ WHICH GUARANTEE A UNIQUE SOLUTION TO (*).



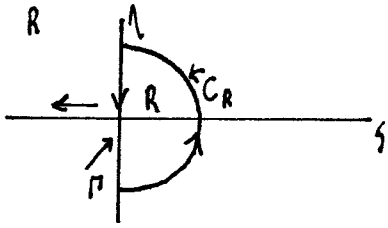
EXAMPLE FIND THE SOLUTION TO

$$u_{xx} + u_{yy} + k^2 u = f(x,y) \quad \text{IN} \quad 0 < x < \infty,\ -\infty < y < \infty$$

WITH $u_x = g(y)$ ON $x = 0$ AND u OUTGOING AS $(x^2+y^2) \to \infty$

APPLY GREEN'S IDENTITY TO THE REGION SHOWN BELOW

$$\iint_R \left[v(\Delta u + k^2 u) - u(\Delta v + k^2 v)\right] d\xi\, d\eta = \int_{C_R} \left(v\, \partial u/\partial n - u\, \partial v/\partial n\right) ds + \int_\Gamma \left(v\, \partial u/\partial n - u\, \partial v/\partial n\right) ds$$



THUS

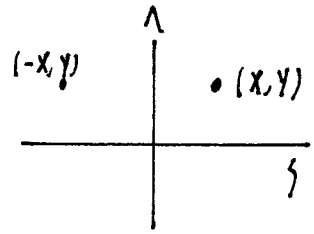
$$\iint_R \left(v[\Delta u + k^2 u] - u[\Delta v + k^2 v]\right) d\xi\, d\eta = \int_{C_R} \left[v(u_r - iku) - u(v_r - ikv)\right]\Big|_{r=R} R\, d\varphi + \int_R^{-R} \left(-v\, \partial u/\partial \xi + u\, \partial v/\partial \xi\right)\Big|_{\xi=0} d\eta$$

NOW $\Delta u + k^2 u = f(\xi,\eta)$, $\Delta v + k^2 v = \delta(\xi - x, \eta - y)$. LET $R \to \infty$ AND IMPOSING OUTGOING CONDITIONS FOR u AND v. ALSO TAKE $v_\xi = 0$ ON $\xi = 0$. THUS, LETTING $R \to \infty$

$$\int_{-\infty}^{\infty}\int_0^{\infty} v f(\xi,\eta)\, d\xi\, d\eta - u(x,y) = \int_{-\infty}^{\infty} v\big|_{\xi=0} g(\eta)\, d\eta$$

$$(*) \qquad u(x,y) = \int_{-\infty}^{\infty}\int_0^{\infty} v f(\xi,\eta)\, d\xi\, d\eta - \int_{-\infty}^{\infty} v\big|_{\xi=0} g(\eta)\, d\eta$$

WHERE v SATISFIES



$$v_{\xi\xi} + v_{\eta\eta} + k^2 v = \delta(\xi - x, \eta - y) \qquad 0 < \xi < \infty,\ -\infty < \eta < \infty$$

$$v_\xi = 0 \text{ ON } \xi = 0 \qquad v \text{ OUTGOING AT } -\infty$$

BY THE METHOD OF IMAGES AND P.37 WE HAVE

$$v = -\frac{i}{4} H_0^{(1)}\left[k\left((\xi - x)^2 + (\eta - y)^2\right)^{1/2}\right] + -\frac{i}{4} H_0^{(1)}\left[k\left((\xi + x)^2 + (\eta - y)^2\right)^{1/2}\right]$$

NOTICE THAT ON $\xi = 0$; $\quad v = -\frac{i}{2} H_0^{(1)}\left[k\left(x^2 + (\eta - y)^2\right)^{1/2}\right]$

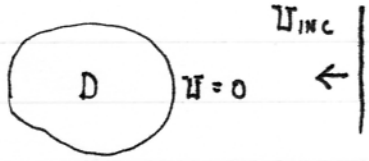
THIS IS TO BE SUBSTITUTED IN $(*)$.

REMARK (i) IT IS CLEAR THAT WE CAN FORMULATE SIMILAR PROBLEMS IN 3-D WHERE THE METHOD OF IMAGES IS NEEDED.

(ii) SOMETIMES SEPARATION OF VARIABLES IS MORE CONVENIENT.

SCATTERING PROBLEMS - WAVE PROPAGATION

CONSIDER THE WAVE EQUATION



$$U_{tt} = c^2 \Delta U \quad \text{OUTSIDE } D$$

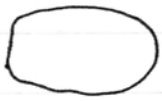
$$(1) \qquad U = 0 \quad \text{ON } \partial D$$

ASSUME THAT FAR FROM THE BODY THERE IS AN INCIDENT PLANE WAVE OF THE FORM $U_{INC} = e^{-ikx - i\omega t}$ WHERE $k = \omega/c$.

NOW WE DECOMPOSE THE TOTAL FIELD U AS

$U = U_s + U_{INC}$ WHERE U_s IS THE SCATTERED FIELD AND IS OUTGOING AT INFINITY. THUS WE WRITE $U_s(\underline{x}, t) = u_s(\underline{x}) e^{-i\omega t}$.

THEN UPON SUBSTITUTING IN (1) WE GET



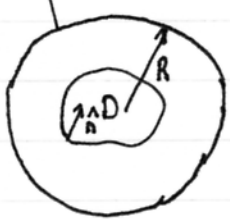
$$Lu_s \equiv \Delta u_s + k^2 u_s = 0 \quad \text{OUTSIDE } D$$

$$(2) \qquad u_s = -e^{-ikx} \quad \text{ON } \partial D$$

$$\partial_r u_s - ik u_s \to 0 \quad \text{AS } r \to \infty \quad \text{OUTGOING CONDITION}$$

TO SOLVE (2) WE IMPOSE GREEN'S IDENTITY TO THE DOMAIN SHOWN BELOW TO GET

$$\iint_{D_R} (v L u_s - u_s L v)\, d\underline{\xi} = \int_{\partial D_R} \left[v(u_{s_R} - ik u_s) - u_s(v_R - ikv) \right] ds + \int_{\partial D} (v \partial_n u_s - u_s \partial_n v)\, ds$$



D_R = sphere of radius R

NOW LET $R \to \infty$ AND IMPOSE OUTGOING CONDITION WHICH ELIMINATES $\int_{\partial D_R}$ TERM. THEN TAKE

$$Lv = \delta(\underline{\xi} - \underline{x}) \quad \underline{x} \text{ OUTSIDE } D$$

$$v = 0 \quad \text{ON } \partial D$$

$$v_r - ikv \to 0 \quad \text{AS } r \to \infty$$

THIS GIVES

(NOTE CHANGE IN SIGN)

$$u_s(\underline{x}) = -\int_{\partial D} u_s \partial_n v\, ds$$

WHERE $\partial_n v$ NOW DENOTES the outward normal derivative as shown

$$u_s = \int_{\partial D} e^{-ikx} \partial_n v\, ds$$