STEADY-STATE DIFFUSION PROBLEMS

THE TIME-DEPENDENT DIFFUSION EQUATION IN A 1-D SLAB DOMAIN $0 < X < L$ IS

$$U_t = [D(X) U_X]_X - q(X) U + F, \quad 0 < X < L$$

$$U(0,t) = U_0, \quad U(L,t) = U_1, \quad U(X,0) = g(X).$$

WHERE
- $D(X) > 0$ IS spatially dependent diffusion coefficient
- $q(X) > 0$ IS bulk desorption.
- F IS the constant source term

THE STEADY-STATE SOLUTION $U_S(X)$ SATISFIES $U_S(X) = \bar{U}(X)$

WHERE

$$(1) \quad \begin{cases} (D(X)\bar{U}')' - q(X)\bar{U} = -F \\ \bar{U}(0) = U_0, \quad \bar{U}(L) = U_1 \end{cases}$$

WE WOULD EXPECT THAT

$$(1.5) \quad \lim_{t \to \infty} U(X,t) = \bar{U}(X).$$

TO EXAMINE THIS QUESTION WE PUT

$$(2) \quad U(X,t) = \bar{U}(X) + e^{-\lambda t}\phi(X).$$

THEN WE OBTAIN THAT

$$(3) \quad \begin{cases} [D(X)\phi']' - q\phi = -\lambda\phi, & 0 < X < L \\ \phi(0) = 0, \quad \phi(L) = 0 \end{cases}$$

NOW (3) IS A STURM-LIOUVILLE PROBLEM

WE WANT TO DETERMINE ALL OF THE EIGENVALUES λ_n AND EIGENFUNCTIONS $\phi_n(x)$ OF (3).

- IF $\lambda_n > 0$ FOR $n = 1, 2, \ldots$

  THEN (1.5) HOLDS.

- WE WRITE (2) BY SUPERPOSITION AS

$$u(x,t) = U(x) + \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \phi_n(x)$$

where $$g(x) = U(x) + \sum_{n=1}^{\infty} c_n \phi_n(x) \longrightarrow c_n = \frac{\int_0^L [g(x) - U(x)]\, \phi_n(x)\, dx}{\int_0^L (\phi_n(x))^2\, dx}$$

- TO ESTIMATE THE SIGN OF λ WE MULTIPLY (3) BY ϕ':

$$\int_0^L \phi \cdot [D(x)\phi']'\, dx - \int_0^L q\phi^2\, dx = -\lambda \int \phi^2\, dx$$

INTEGRATE BY PARTS

$$\phi [D\phi']\Big|_0^L - \int_0^L D\phi'^2 - \int_0^L q\phi^2\, dx = -\lambda \int_0^L \phi^2\, dx .$$

HENCE, $$\lambda \int_0^L \phi^2\, dx = \int_0^L D\phi'^2\, dx + \int_0^L q\phi^2\, dx$$

SINCE $D(x) > 0$, $q(x) > 0$ AND $\phi \neq 0 \longrightarrow \lambda > 0$.

SOLVABLE - STEADY - STATE PROBLEMS

WE NOW CONSIDER A FEW ANALYTICALLY SOLVABLE STEADY-STATE PROBLEMS (1) AND STURM-LIOUVILLE PROBLEMS (3).

EXAMPLE 1 $U'' - \alpha^2 U = 0$

$U(-1) = U_0, \quad U(1) = U_1$

SOLUTION IS $U(x) = A \sinh[\alpha(x+1)] + B \cosh[\alpha(x+1)]$.

$U(-1) = U_0 \longrightarrow U(x) = U_0 \cosh[\alpha(x+1)] + A \sinh[\alpha(x+1)]$

$U(1) = U_1 \longrightarrow U_1 = U_0 \cosh(2\alpha) + A \sinh(2\alpha)$

$$A = \frac{U_1 - U_0 \cosh(2\alpha)}{\sinh(2\alpha)}$$

SO $$U(x) = U_0 \cosh[\alpha(x+1)] + \left(\frac{U_1}{\sinh 2\alpha} - U_0 \coth(2\alpha) \right) \sinh(\alpha(x+1))$$

EXAMPLE 2 $(x^2 U')' - U = 0, \quad 1 < x < 2$

$U(1) = 0, \quad U(2) = 1$

SO $x^2 U'' + 2x U' - U = 0$ EULER EQ.

let $U = x^B \rightarrow B(B-1) + 2B - 1 = 0$

$$B^2 + B - 1 = 0 \quad B_\pm = \frac{-1 \pm \sqrt{5}}{2}$$

SO $U = C_0 x^{B_+} + C_1 x^{B_-}$

BUT $U(1) = 0 \longrightarrow C_0 + C_1 = 0 \longrightarrow U = C_0 (x^{B_+} - x^{B_-})$

$U(2) = 1 \longrightarrow C_0 2^{B_+} + C_1 2^{B_-} = 1$

$$\longrightarrow C_0 = \frac{1}{2^{B_+} - 2^{B_-}}$$

$$\longleftrightarrow U = \frac{1}{2^{B_+} - 2^{B_-}} (x^{B_+} - x^{B_-})$$

EIGENVALUE PROBLEMS IN 1-D

1 CONSIDER $(x^2\phi')' - \phi = -\lambda\phi, \quad 1<x<2; \quad \phi(1) = \phi(2) = 0$

$$x^2\phi'' + 2x\phi' - \phi = -\lambda\phi$$

LET $\phi = x^B \rightarrow B(B-1) + 2B - 1 = -\lambda \rightarrow B^2 + B - 1 + \lambda = 0.$

SO $$B = \frac{-1 \pm \sqrt{1-4(\lambda-1)}}{2} = \frac{-1 \pm \sqrt{5-4\lambda}}{2} = \frac{-1 \pm i\sqrt{4\lambda-5}}{2}$$

SO $\phi = x^{-1/2} \sin\left(\frac{\sqrt{4\lambda-5}}{2} \ln x\right)$ SATISFIES $\phi(1) = 0$

NOW $\phi(2) = 0 \rightarrow \frac{\sqrt{4\lambda-5}}{2} \ln 2 = n\pi, \quad n = 1, 2, 3, \ldots$

SO $$4\lambda - 5 = \frac{4n^2\pi^2}{(\ln 2)^2}$$

$$\lambda_n = \frac{n^2\pi^2}{(\ln 2)^2} + \frac{5}{4}, \quad n = 1, 2, 3, \ldots$$

X 2 PUT THE FOLLOWING PROBLEM IN STURM LIOUVILLE FORM

$$\psi'' - x\psi' - \psi = -\lambda\psi, \quad \psi(0) = \psi(1) = 0, \quad 0 < x < 1$$

NOW $e^{\int -x\,dx} = e^{-x^2/2}$

SO $$e^{-x^2/2}(\psi'' - x\psi') - e^{-x^2/2}\psi = -\lambda e^{-x^2/2}\psi$$

STURM-LIOUVILLE FORM $$\frac{d}{dx}\left(e^{-x^2/2}\psi'\right) - e^{-x^2/2}\psi = -\lambda e^{-x^2/2}\psi$$

THE WEIGHT FUNCTION IS $w(x) = e^{-x^2/2}$.

SO $\frac{d}{dx}[D(x)\psi'] - e^{-x^2/2}\psi = -\lambda w(x)\psi$

RADIALLY-SYMMETRIC PROBLEMS

IN A SPHERE OF RADIUS R THE STEADY-STATE DIFFUSION EQUATION IS OFTEN OF THE FORM

EX1 $\frac{1}{r^2}(r^2 U')' - \alpha^2 U = F$, $0 < r < L$, F = CONSTANT.

$U(0)$ = bounded $\quad U(L) = 0$

SO $U'' + \frac{2U'}{r} - \alpha^2 U = F$

$U(0)$ NICE, $U(L) = 0$

let $U(r) = V(r)/r$ $\quad U = r^{-1}V$

$U_r = -r^{-2}V + r^{-1}V_r$

$U_{rr} = 2r^{-3}V - 2r^{-2}V_r + r^{-1}V_{rr}$

THIS YIELDS THAT,

$$2r^{-3}V - 2r^{-2}V_r + r^{-1}V_{rr} + 2r^{-2}V_r - 2r^{-3}V - \alpha^2 r^{-1}V = F.$$

HENCE $V_{rr} - \alpha^2 V = rF$

THE HOMOGENEOUS SOLUTION IS $V = A\sinh(\alpha r) + B\cosh(\alpha r)$.

FOR THE PARTICULAR SOLUTION LET $V = Cr^{\beta}$. THEN $-\alpha^2 C = F$

THIS YIELDS THAT

$$U = \frac{A}{r}\sinh(\alpha r) + \frac{B}{r}\cosh(\alpha r) - \frac{F}{\alpha^2} r$$

$U(0)$: NICE $\longrightarrow$ $B = 0$

$$U(L) = 0 \longrightarrow \frac{A}{L}\sinh(\alpha L) = \frac{FL}{\alpha^2} \longrightarrow A = \frac{FL^2}{\alpha^2 \sinh(\alpha L)}$$

SO $$U = \frac{FL^2}{\alpha^2 r \sinh(\alpha L)}\sinh(\alpha r) - \frac{F}{\alpha^2} r.$$

EX 2 FOR $U'' + \frac{2}{r}U' = F$ IN $0 < r < L$

$U(0)$: NICE $\quad U(L) = 1$

HOMOGENEOUS PROBLEM $\quad r U'' + 2U' = 0$

$$r^2 U'' + 2r U' = 0$$

$$(r^2 U')' = 0 \rightarrow r^2 U' = C.$$

SO $$U' = \frac{C}{r^2} \longrightarrow U = -\frac{C}{r} + d.$$

IF $U(0)$: NICE $\longrightarrow C = 0. \longrightarrow U_{homog} = D$

FOR PARTICULAR SOLUTION LET $U = Br^2$.

THEN $$2B + \frac{2}{r} 2Br = F \longrightarrow B = F/6.$$

SO $$U_{particular} = Fr^2/6.$$

SO $$U = D + Fr^2/6. \quad U(L) = 1 \longrightarrow D = -\frac{FL^2}{6} + 1$$

$$U = 1 + \frac{F}{6}(r^2 - L^2).$$

EIGENVALUE PROBLEM IN A SPHERE

$$\phi'' + \frac{2}{r}\phi' = -\lambda\phi \quad , \quad \phi(0): \text{NICE}, \quad \phi(L) = 0.$$

NOW $\quad r^2\phi'' + 2r\phi' = -\lambda r^2\phi$

$$(r^2\phi')' = -\lambda r^2\phi \longrightarrow \int_0^L r^2 \phi_n \phi_m \, dr = 0 \quad n \neq m.$$

orthogonality relation.

$\uparrow$ weight

Let $\quad \phi = \frac{1}{r}\psi.$

Then we get a) in EX 1 on previous page

$$\psi'' + \lambda\psi = 0 \qquad \psi = A\sin(\sqrt{\lambda}r) + B\cos(\sqrt{\lambda}r)$$

so $\quad \phi = \frac{A}{r}\sin(\sqrt{\lambda}r) + \underbrace{\frac{B}{r}\cos(\sqrt{\lambda}r)}$

unbounded as $r \to 0$.

THUS $\quad \phi(L) = 0 \longrightarrow \sin(\sqrt{\lambda}L) = 0 \qquad \sqrt{\lambda}L = n\pi, \quad n = 1, 2, \ldots$

$$\lambda_n = n^2\pi^2/L^2 \qquad \phi_n(r) = \frac{1}{r}\sin\left(n\pi r/L\right).$$

RADIALLY SYMMETRIC STEADY-STATE DIFFUSION IN A CYLINDER

X1 THE STEADY-STATE PROBLEM IS OFTEN

$$U_{rr} + \frac{1}{r} U_r = F \quad , \quad a < r < b$$

$$U(a) = U_0, \quad U(b) = U_1. \qquad F = \text{CONSTANT}$$

THE HOMOGENEOUS PROBLEM IS

$$U_{rr} + \frac{1}{r} U_r = 0 \rightarrow (r U_r)_r = 0 \rightarrow r U_r = A$$

$$\text{so} \quad U_r = \frac{A}{r} \quad \text{OR} \quad \underset{\text{homog}}{U} = A \log r + B$$

THE PARTICULAR SOLUTION U_{part} HAS THE FORM

$$U_{part} = C r^2 \rightarrow 2C + 2C = F \rightarrow C = F/4$$

$$\text{so} \quad U = A \log r + B + \frac{F}{4} r^2$$

$$\text{NOW} \quad U(a) = U_0 \rightarrow U = A \log(r/a) + \frac{F}{4}(r^2 - a^2) + U_0$$

$$\text{NOW} \quad U(b) = U_1 \rightarrow A \log(b/a) = U_1 - U_0 + \frac{F}{4}(b^2 - a^2) \rightarrow \text{determine } A$$

$$A = \frac{U_1 - U_0 + F/4 (b^2 - a^2)}{\log(b/a)}$$

EX 2 NOW CONSIDER THE ODE FOR $y = y(x)$

$$y'' + \frac{1}{x} y' - y = 0 \qquad \text{OR} \qquad x y'' + y' - x y = 0$$

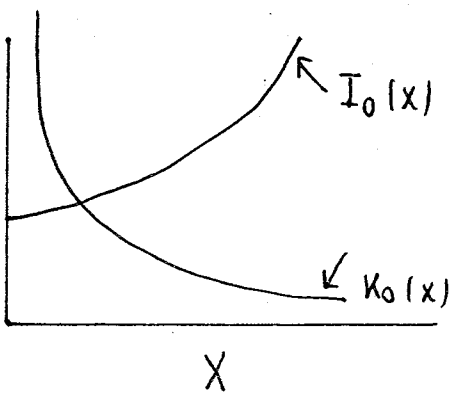
THE TWO LINEARLY INDEPENDENT SOLUTIONS ARE

$Y_1 = I_0(x)$ MODIFIED BESSEL FUNCTION OF THE FIRST KIND OF ORDER 0

$Y_2 = K_0(x)$ " " " " " SECOND KIND " " "

THESE FUNCTIONS CAN BE DEFINED FOR ALL X BY POWER SERIES - type solutions. FROBENIUS SERIES.

THEY HAVE THE BEHAVIORS AS SHOWN



$I_0(x)$ BOUNDED AS $X \to 0$

UNBOUNDED AS $X \to \infty$

$K_0(x)$ BOUNDED AS $X \to \infty$

UNBOUNDED AS $X \to 0$

WE HAVE THE BEHAVIOR

$$I_0(x) \sim 1 + X^2/2^2 + \cdots \quad \text{AS} \quad X \to 0$$

$$I_0(x) \sim e^X/\sqrt{2\pi X} \quad \text{AS} \quad X \to \infty$$

$$K_0(x) \sim -\log X \quad \text{AS} \quad X \to 0$$

$$K_0(x) \sim \sqrt{\frac{\pi}{2X}}\, e^{-X} \quad \text{AS} \quad X \to \infty.$$

EX1 FIND THE STEADY-STATE SOLUTION TO

$$U'' + \frac{1}{X} U' - B^2 U = F \quad \text{IN} \quad 0 < X < R; \quad U(0) \text{ NICE} \to \text{bounded}$$
$$U(1) = U_1.$$

THE PARTICULAR SOLUTION IS $U_{part} = -F/B^2$.

SO $U = C_1 I_0(BX) + C_2 K_0(BX) - F/B^2$

K_0 UNBOUNDED AS $X \to 0 \longrightarrow C_2 = 0$

SO $U = C_1 I_0(BX) - F/B^2$.

NOW $U(1) = U_1 \longrightarrow U_1 + F/B^2 = C_1 I_0(B) \longrightarrow$ gives C_1

$$U(X) = \left(U_1 + F/B^2\right) \frac{I_0(BX)}{I_0(B)} - \frac{F}{B^2}.$$

EX 2 SOLVE $U'' + \frac{1}{X} U' - B^2 U = 0, \quad X \geq 1$

$U(1) = U_1$ $U(\infty)$ BOUNDED.

THE SOLUTION IS $U = C_1 I_0(BX) + C_2 K_0(BX)$

$U(\infty)$ BOUNDED $\longrightarrow C_1 = 0$ (SINCE I_0 UNBOUNDED AT ∞)

$U(1) = U_1 \longrightarrow C_2 K_0(B) = U_1$

SO $U = U_1 \frac{K_0(BX)}{K_0(B)}$