DERIVATION OF THE HEAT EQUATION

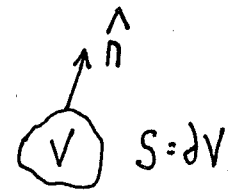
LET V BE AN ARBITRARILY SMALL DOMAIN INSIDE A GIVEN VOLUME Ω

rate of change of heat energy = heat flowing across boundaries per unit time + heat energy generated internally per unit time

heat energy $= \int_V c\rho u \, d\underline{x}$ WITH c = specific heat, ρ = density, $u = u(\underline{x}, t)$ temperature

heat energy generated internally per unit time $= \int_V Q(\underline{x}, t, u) \, d\underline{x}$

LET $\underline{\phi}$ = heat FLUX VECTOR ON ∂V. (boundary of V).



THE NORMAL COMPONENT ON V IS $\underline{\phi} \cdot \hat{\underline{n}}$

IF $\underline{\phi} \cdot \hat{n} < 0$ heat FLUX IS INWARD TO V

THEN
$$\frac{d}{dt} \int_V c\rho u \, d\underline{x} = -\int_S \underline{\phi} \cdot \hat{\underline{n}} \, dS + \int_V Q \, d\underline{x}$$

NOW USE THE DIVERGENCE THEOREM $\int_V \nabla \cdot \underline{A} \, d\underline{x} = \int_S \underline{A} \cdot \hat{\underline{n}} \, dS$.

THUS
$$\frac{d}{dt} \int_V c\rho u \, d\underline{x} + \int_V \nabla \cdot \underline{\phi} \, dx - \int_V Q \, d\underline{x} = 0.$$

NOW IF c, ρ INDEPENDENT OF t WE HAVE
$$\int_V \left(c\rho u_t + \nabla \cdot \underline{\phi} - Q \right) d\underline{x} = 0.$$

SINCE V IS ARBITRARY DOMAIN WE HAVE

$$c\rho u_t = -\nabla\cdot\underline{\phi} + Q(\underline{x},t,u).$$

FINALLY BY FOURIER'S LAW (OF FICK'S LAW OF DIFFUSION)

$$\underline{\phi} = -K\nabla u. \qquad K: \text{thermal conductivity}.$$

THIS LEADS TO $\quad u_t = D\nabla^2 u + Q(\underline{x},t,u)$

WITH $Q(\underline{x},t,u) = Q/\rho c. \qquad D = K/\rho c$ thermal diffusivity

WE THEN HAVE THE BOUNDARY CONDITION THAT

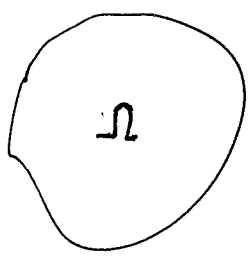
$$\underline{\phi}\cdot\hat{n} = h(u-u_b) \qquad h>0 \quad h = \text{CONSTANT}.$$

THEN WE WRITE $\underline{\phi} = -K\nabla u$ SO THAT NEWTON'S LAW OF COOLING IS

$$\nabla u\cdot\hat{n} = \partial_n u = -b(u-u_b) \quad \text{ON } \partial\Omega \qquad b = h/K \text{ BIOT NUMBER},$$

WITH u_b GIVEN, IS THE OUTSIDE TEMPERATURE.

THIS GIVES THE HEAT CONDUCTION PROBLEM



$$u_t = D\nabla^2 u + Q(\underline{x},t,u) \quad \text{IN } \Omega$$

$$\partial_n u = -b(u-u_b) \quad \text{ON } \partial\Omega \quad \left(\begin{array}{c}\text{BOUNDARY}\\ \text{CONDITION}\end{array}\right)$$

$$u(\underline{x},0) = u_0(\underline{x}) \quad \text{INITIAL TEMPERATURE}.$$

<u>REMARK</u> (i) $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$ WHEN $\underline{x} = (x,y,z)$.

OCASSIONALLY ONE WRITES $\nabla^2 = \Delta$.

(ii) WE CAN ONLY SOLVE LINEAR PROBLEMS WHERE

$$Q(\underline{x},t,u) = b(\underline{x},t)\,u + F(\underline{x},t);$$

IF $b<0$ SINK OF HEAT PROPORTIONAL TO u.

$F<0$ CORRESPONDS TO SINK OF HEAT.

THE ONE-DIMENSIONAL PROBLEM WITH $Q(\underline{x}, t, u) = F(\underline{x})$

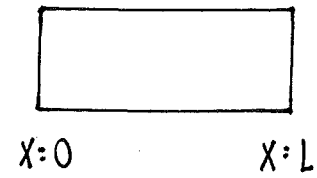
IS SIMPLY

$$u_t = D u_{xx} + F(x) \qquad 0 \le x \le L$$

$$u_x = -h_2 (u - u_2) \quad \text{ON } x = L$$

$$u_x = h_1 (u - u_1) \quad \text{ON } x = 0$$

$$u(x, 0) = u_0(x)$$



THE STEADY-STATE SOLUTION CORRESPONDING TO

$$u_s(x) = \lim_{t \to \infty} u(x, t)$$

SATISFIES

$$\begin{cases} D u_s'' = -F(x) & 0 < x < L \\ u_s' = -h_2 (u_s - u_2) & \text{ON } x = L; \quad u_s' = h_1 (u_s - u_1), \ x = 0 \end{cases}$$

<u>REMARK</u> (i) FOR RADIALLY SYMMETRIC DIFFUSION IN A <u>CIRCLE</u> THEN $\nabla^2 u = u_{rr} + \frac{1}{r} u_r$

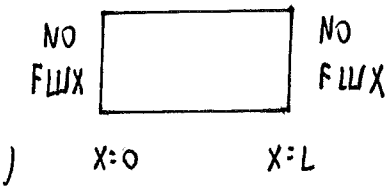
(ii) FOR RADIALLY SYMMETRIC DIFFUSION IN A <u>SPHERE</u> THEN $\nabla^2 u = u_{rr} + \frac{2}{r} u_r$.

(iii) TO SOLVE $u_{rr} + \frac{(n-1)}{r} u_r = F_0$ (F_0: CONSTANT), LET $u = A r^2$ TO OBTAIN $2A + 2(n-1)A = F_0$ SO THAT $A = F_0 / 2n$

(iii) IN A CIRCLE $\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$.

(iv) SUPPOSE $u_t = [D u_x]_x$ THEN FOR THE STEADY-STATE SOLUTION WE REQUIRE $D_+ u_x|_{x_0^+} = D_- u_x|_{x_0^-}$ WHEN $D = \begin{cases} D_+, & x > x_0 \\ D_-, & x < x_0 \end{cases}$.

INSULATING BOUNDARY CONDITIONS



CONSIDER
$$(*)\begin{cases} u_t = D u_{xx} & 0<x<L,\ t>0 \\ u_x = 0 \quad \text{ON } x=0, L; & u(x,0) = f(x) \end{cases}$$

THE STEADY-STATE SOLUTION IS

$$u_s'' = 0,\ 0<x<L; \qquad u_s'(0) = u_s'(L) = 0.$$

THIS GIVES $u_s = A$ $\quad A$ = CONSTANT TO BE FOUND.

WE WILL SHOW THAT $A = \frac{1}{L}\int_0^L f(x)\,dx$ = average of initial temperature.

TO SHOW THIS WE INTEGRATE

$$\int_0^L u_t\,dx = \int_0^L D u_{xx}\,dx = D u_x\Big|_0^L = 0.$$

HENCE $\frac{d}{dt}\int_0^L u(x,t)\,dx = 0 \longrightarrow \int_0^L u(x,t)\,dx = C$ CONSTANT

FOR ALL TIME. EVALUATE AT $t=0$ TO OBTAIN $\int_0^L u(x,t)\,dx = \int_0^L f(x)\,dx$

THEN LET $t\to\infty$ AND USE $u(x,t)\to A$ AS $t\to\infty$

SO THAT $\int_0^L A\,dx = AL = \int_0^L f(x)\,dx \longrightarrow A = \frac{1}{L}\int_0^L f(x)\,dx.$

IN THE MULTI-DIMENSIONAL CASE $u_t = D\nabla^2 u$ IN Ω

WITH $\partial_n u = 0$ ON $\partial\Omega$ AND $u(\underline{x},0) = f(x)$ YOU WILL SHOW THAT

$$\lim_{t\to\infty} u(\underline{x},t) = \frac{1}{\mathrm{VOL}(\Omega)}\int_\Omega f(\underline{x})\,d\underline{x}.$$

ALTERNATIVELY FOR (*) IN 1-DIMENSION WE WRITE

$$W(x,t) = X(x)\,T(t) \qquad \frac{T'}{DT} = \frac{X''}{X} = -\lambda.$$

THUS $\quad \phi'' + \lambda\phi = 0 \quad \longrightarrow \quad \lambda_0 = 0, \ \phi_0 = 0$

$\phi'(0) = \phi'(L) = 0 \qquad \lambda_n = \dfrac{n^2\pi^2}{L^2}, \ \phi_n(x) = \cos\left(\dfrac{n\pi x}{L}\right)$

$n = 1, 2, 3, \ldots$

THUS $\quad W(x,t) = C_0 + \displaystyle\sum_{n=1}^{\infty} C_n e^{-\lambda_n D t} \cos\left(\frac{n\pi x}{L}\right)$

THEN $\quad f(x) = C_0 + \displaystyle\sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right)$

INTEGRATING FROM $x = 0, L \quad \longrightarrow \displaystyle\int_0^L f(x)\,dx = C_0 L$

SO $\quad C_0 = \dfrac{1}{L}\displaystyle\int_0^L f(x)\,dx.$

AS $t \to \infty$ THEN $\displaystyle\lim_{t\to\infty} W(x,t) = C_0 = \frac{1}{L}\int_0^L f(x)\,dx.$

STURM-LIOUVILLE PROBLEMS

THE SL PROBLEM HAS THE FORM

$$(p(x)\phi')' - q(x)\phi + \lambda w(x)\phi = 0 \qquad 0<x<1 \qquad w(x)>0, p(x)>0$$
$$\phi'(0) - h_1\phi(0) = 0, \quad \phi'(1) + h_2\phi(1) = 0. \qquad \text{ON } 0<x<1$$

WE WRITE THE EIGENVALUE PROBLEM AS

$$\text{STURM-LIOUVILLE } (*) \begin{cases} \mathcal{L}\phi = \lambda w(x)\phi & \mathcal{L}\phi = -(p(x)\phi')' + q(x)\phi \\ \phi'(0) - h_1\phi(0) = 0, & \phi'(1) + h_2\phi(1) = 0 \end{cases} \qquad w(x)>0, \; p(x)>0 \text{ ON } 0<x<1$$

WE WILL DERIVE MANY PROPERTIES OF $(*)$. TO DO SO WE FIRST DERIVE LAGRANGE'S IDENTITY:

$$(1) \quad \int_0^1 (v\mathcal{L}u - u\mathcal{L}v)\,dx = -p(x)u'v\Big|_0^1 + p(x)uv'\Big|_0^1.$$

<u>PROOF</u> WE WRITE
$$\int_0^1 v\mathcal{L}u\,dx = \int_0^1 [-v(pu')' + vqu]\,dx$$
$$= -pu'v\Big|_0^1 + \int_0^1 ((pv')u' + vqu\,dx)$$
$$= -pu'v\Big|_0^1 + pv'u\Big|_0^1 - \int_0^1 [(pv')'u - vqu]\,dx$$

THIS YIELDS THAT
$$\int_0^1 v\mathcal{L}u\,dx = -pu'v\Big|_0^1 + pv'u\Big|_0^1 + \int_0^1 u\mathcal{L}v\,dx \qquad \square$$

NOW SUPPOSE THAT u, v SATISFY THE BOUNDARY CONDITIONS IN $(*)$ SO THAT $u'(0) - h_1u(0) = 0$, $u'(1) + h_2u(1) = 0$, $v'(0) - h_1v(0) = v'(1) + h_2v(1) = 0$. THEN WE CAN ADD AND SUBTRACT IN (1):

$$\int_0^1 (v\mathcal{L}u - u\mathcal{L}v)\,dx = p(1)u(1)\big(v'(1) + h_2v(1)\big) - p(1)v(1)\big(u'(1) + h_2u(1)\big)$$
$$+ p(0)v(0)\big(u'(0) - h_1u(0)\big) - p(0)u(0)\big(v'(0) - h_1v(0)\big)$$
$$= 0$$

THEREFORE, WE OBTAIN

$$\int_0^1 v\,\mathcal{L}u\,dx = \int_0^1 u\,\mathcal{L}v\,dx \quad \text{WHENEVER } u, v \text{ SATISFY THE B.C.}$$

IF WE DEFINE $(a, b) \equiv \int_0^1 a\,b\,dx$ THEN WE CAN WRITE $(v, \mathcal{L}u) = (u, \mathcal{L}v)$

WE NOW DERIVE (OR STATE) MANY OF THE KEY PROPERTIES OF THE STURM-LIOUVILLE PROBLEM.

<u>PROPERTIES</u>

(i) THE EIGENVALUES HAVE THE PROPERTIES

a) λ IS REAL

b) THERE ARE AN INFINITE NO. OF EIGENVALUES λ_j WITH $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ AND $\lambda_j \to +\infty$ AS $j \to \infty$.

c) $\lambda_j > 0$ WHEN $h_1 \geq 0$ AND $h_2 \geq 0$, AND $q(x) \geq 0$, AND $h_1, h_2 > 0$

(ii) THE EIGENFUNCTIONS $y = \phi_j(x)$ FOR $j = 1, 2, 3, \dots$ HAVE THE PROPERTIES

a) $\phi_j(x)$ ARE REAL AND CAN BE NORMALIZED $\int_0^1 w\phi_j^2\,dx = 1$

b) $\int_0^1 \phi_j(x)\,\phi_k(x)\,w(x)\,dx = 0 \qquad j \neq k$

(iii) EXPANJION PROPERTY

ANY FUNCTION $f(x)$ WITH $\int_0^1 (f(x))^2\,dx < \infty$ CAN BE EXPANDED AS

$$f(x) = \sum_{n=1}^{\infty} c_n\,\phi_n(x)$$

WHERE BY ORTHOGONALITY

$$c_n = \frac{\int_0^1 f(x)\,\phi_n(x)\,w(x)\,dx}{\int_0^1 (\phi_n(x))^2\,w(x)\,dx}$$

PROOF OF SOME OF THE PROPERTIES

(i) EIGENVALUES ARE REAL

LET $\mathcal{L}\phi = \lambda w \phi$ WITH $\phi'(0) - h_1 \phi(0) = 0$ AND $\phi'(1) + h_2 \phi(1) = 0$.

TAKE THE CONJUGATE $\mathcal{L}\bar{\phi} = \bar{\lambda} w \bar{\phi}$. NOW USE LAGRANGE'S IDENTITY

$$\int_0^1 (\phi \mathcal{L}\bar{\phi} - \bar{\phi}\mathcal{L}\phi)\,dx = 0 \longrightarrow (\phi, \mathcal{L}\bar{\phi}) - (\bar{\phi}, \mathcal{L}\phi) = 0.$$

THIS YIELDS THAT $0 = (\phi, \bar{\lambda}\bar{\phi}) - (\bar{\phi}, \lambda\phi) = (\bar{\lambda} - \lambda)(\bar{\phi}, \phi) = (\bar{\lambda} - \lambda)\int_0^1 w\phi\bar{\phi}\,dx$

HENCE $(\bar{\lambda} - \lambda)\int_0^1 w|\phi|^2\,dx = 0 \rightarrow \lambda = \bar{\lambda} \rightarrow \lambda$ IS REAL.

(ii) SHOW $\lambda_j > 0$ WHEN $h_1 \geq 0$ AND $h_2 \geq 0$, $q(x) \geq 0$, $h_1 h_2 > 0$

WE WRITE $\mathcal{L}\phi = \lambda w \phi$

MULTIPLY BY ϕ AND INTEGRATE $\int_0^1 \phi \mathcal{L}\phi\,dx = \lambda \int_0^1 w\phi^2\,dx$

NOW INTEGRATE BY PARTS: $-p(x)\phi'(x)\phi(x)\Big|_0^1 + \int_0^1 (p\phi'^2 + q\phi^2)\,dx = \lambda\int_0^1 w\phi^2\,dx.$

NOW $\phi'(1) = -h_2\phi(1)$ AND $\phi'(0) = h_1\phi(0)$, WHICH YIELDS

$$p(1)h_2(\phi(1))^2 + p(0)h_1(\phi(0))^2 + \int_0^1 (p\phi'^2 + q\phi^2)\,dx = \lambda\int_0^1 w\phi^2\,dx.$$

NOW SINCE $q(x) \geq 0$ (BY ASSUMPTION) AND $p(x) > 0$, $w(x) > 0$ ON $0 < x < 1$ FOR STURM-LIOUVILLE THEN WE HAVE $\lambda > 0$.

REMARK IF $q \equiv 0$ FOR $0 \leq x \leq 1$ AND $h_1 = h_2 = 0$ THEN WE HAVE $(p(x)\phi')' + \lambda w(x)\phi = 0$ WITH $\phi'(0) = \phi'(1) = 0$.

THIS HAS AN EIGENVALUE $\lambda = 0$ AND EIGENFUNCTION $\phi = 1$.

(ii) EIGENFUNCTIONS CORRESPONDING TO DIFFERENT EIGENVALUES ARE ORTHOGONAL

WE WRITE $\mathcal{L}\phi_j = \lambda_j w \phi_j$ WITH ϕ_j AND ϕ_k

$\mathcal{L}\phi_k = \lambda_k w \phi_k$ SATISFYING THE BOUNDARY CONDITIONS.

THEN $(\phi_k, \mathcal{L}\phi_j) = (\phi_j, \mathcal{L}\phi_k)$ BY LAGRANGE'S IDENTITY.

THIS YIELDS $\lambda_j(\phi_k, w\phi_j) = \lambda_k(\phi_k, w\phi_j)$.

THEREFORE $(\lambda_j - \lambda_k)\int_0^1 w\phi_j\phi_k\, dx = 0$.

IF $\lambda_j \neq \lambda_k$ THEN $\int_0^1 w\phi_j\phi_k\, dx = 0 \rightarrow$ ORTHOGONALITY.

(iii) IT IS DIFFICULT TO PROVE THAT ANY $f(x)$ WITH $\int_0^1 (f(x))^2 dx < \infty$ CAN BE EXPANDED AS $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$.

TO DETERMINE THE COEFFICIENTS, MULTIPLY BY $w(x)\phi_m(x)$ AND GET

$$\int_0^1 f(x)\phi_m(x)w(x)\,dx = \sum_{n=1}^{\infty} c_n \int_0^1 \phi_m(x)\phi_n(x)w(x)\,dx$$

BY ORTHOGONALITY $c_m \int_0^1 (\phi_m(x))^2 w(x)\,dx = \int_0^1 f(x)\phi_m(x)w(x)\,dx$,

WHICH DETERMINES c_m.

EXAMPLE 1 FIND THE NORMALIZED EIGENFUNCTIONS FOR

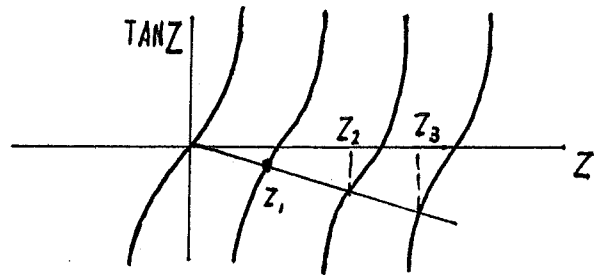
$$\phi'' + \lambda\phi = 0, \quad 0 < x < 1$$
$$\phi(0) = 0, \quad \phi'(1) + \phi(1) = 0$$

WE KNOW THAT $\lambda > 0$ SINCE $h_1 h_2 > 0$ (property ii).

WE OBTAIN $\phi = A\sin(\sqrt{\lambda}\,x)$. NOW $A\sqrt{\lambda}\cos(\sqrt{\lambda}) + A\sin(\sqrt{\lambda}) = 0$.

THUS λ SATISFIES $\tan[\sqrt{\lambda}] = -\sqrt{\lambda}$.

WITH $\lambda > 0$. PLOT $\tan z = -z$.



THERE ARE AN INFINITE # OF ROOTS WITH $z_n \approx \frac{(2n+1)}{2}\pi$ AS $n \to \infty$.

NEXT WE WRITE $\phi_n(x) = A_n \sin(\sqrt{\lambda_n}\,x)$. THEN TO NORMALIZE WE WRITE $\int_0^1 A_n^2 \sin^2(\sqrt{\lambda_n}\,x)\,dx = 1 = A_n^2 \int_0^1 \left(\frac{1-\cos(2\sqrt{\lambda_n}\,x)}{2}\right)dx$

THIS YIELDS $\frac{A_n^2}{2}\left[1 - \frac{1}{2\sqrt{\lambda_n}}\sin(2\sqrt{\lambda_n})\right] = 1$

OR $\frac{A_n^2}{2}\left[1 - \frac{\sin(\sqrt{\lambda_n})\cos(\sqrt{\lambda_n})}{\sqrt{\lambda_n}}\right] = \frac{A_n^2}{2}\left[1 + \cos^2(\sqrt{\lambda_n})\right] = 1 \rightarrow A_n = \frac{\sqrt{2}}{[1+\cos^2(\sqrt{\lambda_n})]}$

(HERE WE USED $\sin(\sqrt{\lambda_n}) = -\sqrt{\lambda_n}\cos(\sqrt{\lambda_n})$ FROM THE EIGENVALUE RELATION).

THIS YIELDS, $\phi_n(x) = \frac{\sqrt{2}}{[1+\cos^2(\sqrt{\lambda_n})]^{1/2}}\sin(\sqrt{\lambda_n}\,x)$

NOW IF WE EXPAND $f(x) = x$ AS $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$. WE HAVE

$$c_n = \int_0^1 x\,\phi_n(x)\,dx = A_n \int_0^1 x\sin(\sqrt{\lambda_n}\,x)\,dx = A_n\left[\frac{-x}{\sqrt{\lambda_n}}\cos(\sqrt{\lambda_n}\,x)\Big|_0^1 + \frac{1}{\sqrt{\lambda_n}}\int_0^1 \cos(\sqrt{\lambda_n}\,x)\,dx\right]$$

THIS YIELDS THAT $c_n = A_n\left[-\frac{1}{\sqrt{\lambda_n}}\cos(\sqrt{\lambda_n}) + \frac{1}{\lambda_n}\sin(\sqrt{\lambda_n})\right]$

NOW REPLACE $\cos(\sqrt{\lambda_n}) = -\frac{1}{\sqrt{\lambda_n}} \sin(\sqrt{\lambda_n})$, WHICH GIVES $C_n = \frac{2}{\lambda_n} \sin(\sqrt{\lambda_n}) A_n$.

THE FINAL RESULT IS $f(x) = x = \sum_{n=1}^{\infty} \frac{2}{\lambda_n} \sin(\sqrt{\lambda_n}) A_n^2 \sin(\sqrt{\lambda_n}\, x)$

FINALLY $$f(x) = x = 4 \sum_{n=1}^{\infty} \frac{\sin(\sqrt{\lambda_n})}{\lambda_n (1 + \cos^2(\sqrt{\lambda_n}))} \sin(\sqrt{\lambda_n}\, x).$$

USE THIS EXPANSION TO SOLVE THE HEAT CONDUCTION PROBLEM GIVEN BY

$$\begin{cases} u_t = \alpha^2 u_{xx}, & 0 < x < 1 \\ u(0,t) = 1, & u_x(1,t) + u(1,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

WE FIRST CALCULATE THE STEADY-STATE SOLUTION $u_s(x)$ WHICH SATISFIES

$$u_s'' = 0$$

$$u_s(0) = 1, \quad u_s'(1) + u_s(1) = 0.$$

WE GET $u_s(x) = Ax + B$ THEN $u_s(0) = 1$ GIVES $u_s(x) = Ax + 1$.

THIS YIELDS $A + (A+1)$ OR $A = -1/2$. $\longrightarrow$ $u_s(x) = -x/2 + 1$.

FINALLY, WE WRITE $u = u_s + v$. THIS YIELDS THAT

$$(*) \begin{cases} v_t = \alpha^2 v_{xx} \\ v(0,t) = 0, \quad v_x(1,t) + v(1,t) = 0 \\ v(x,0) = f(x) - u_s(x) \end{cases}$$

WE WRITE $v = XT$ SO THAT $\frac{T'}{\alpha^2 T} = \frac{X''}{X} = -\lambda$

$T' = -\alpha^2 \lambda T$

$X'' + \lambda X = 0$

$T = e^{-\alpha^2 \lambda t}$

THIS YIELDS THE EIGENVALUE PROBLEM

$$\phi'' + \lambda\phi = 0, \qquad 0 < x < 1$$

$$\phi(0) = 0, \quad \phi'(1) + \phi(1) = 0$$

THE NORMALIZED EIGENFUNCTIONS ARE $\phi_n(x) = \dfrac{\sqrt{2}}{[1+\cos^2(\sqrt{\lambda_n})]^{1/2}} \sin(\sqrt{\lambda_n}\, x)$

WITH $\tan(\sqrt{\lambda_n}) = -\sqrt{\lambda_n}$. WE EXPAND

$$v(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2 \lambda_n t} \phi_n(x)$$

NOW $\quad v(x,0) = f(x) - u_s(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$

BY ORTHOGONALITY $\quad c_n = \int_0^1 (f(x) - u_s(x))\, \phi_n(x)\, dx.$

AND THEN $\quad u(x,t) = u_s(x) + \sum_{n=1}^{\infty} c_n e^{-\alpha^2 \lambda_n t} \phi_n(x).$

FOR $t \to \infty$, $\quad u(x,t) \sim u_s(x) + c_1 e^{-\alpha^2 \lambda_1 t} \phi_1(x) + \cdots$

<u>EXAMPLE</u> FIND THE NORMALIZED EIGENFUNCTIONS FOR

$$\begin{cases} (x^2\phi')' + \lambda\phi = 0, & 1 < x < 2 \\ \phi(1) = 0, \quad \phi(2) = 0 \end{cases}$$

WE EXPAND OUT TO OBTAIN $x^2\phi'' + 2x\phi' + \lambda\phi = 0$.

WE LET $\phi = x^r$ TO OBTAIN $r(r-1) + 2r + \lambda = 0$.

THIS YIELDS $r = \dfrac{-1 \pm \sqrt{1-4\lambda}}{2}$.

FOR AN EIGENVALUE WE NEED $1 - 4\lambda < 0$ OR $\lambda > 1/4$.

THIS YIELDS $r = -\dfrac{1}{2} \pm \dfrac{i}{2}\sqrt{4\lambda - 1}$.

OUR SOLUTION IS

$$\phi(x) = c_1 x^{-1/2} \sin\left(\frac{\sqrt{4\lambda-1}}{2}\log x\right) + c_2 x^{-1/2}\cos\left(\frac{\sqrt{4\lambda-1}}{2}\log x\right)$$

NOW $\phi(1) = 0$ GIVES $c_2 = 0$.

$\phi(2) = 0$ GIVES $\sin\left(\dfrac{\sqrt{4\lambda-1}}{2}\log 2\right) = 0$ OR $\dfrac{\sqrt{4\lambda-1}}{2}\log 2 = n\pi$

$n = 1, 2, 3, \ldots$

THIS YIELDS $\lambda_n = \dfrac{1}{4} + \dfrac{n^2\pi^2}{(\log 2)^2}$ $\quad n = 1, 2, 3, \ldots$

AND $\phi_n(x) = c\, x^{-1/2} \sin\left(\dfrac{n\pi \log x}{\log 2}\right)$

THEN WE NORMALIZE WITH $\int_1^2 \phi_n^2(x)\,dx = 1$. THIS YIELDS THAT

$$c^2 \int_1^2 \frac{1}{x} \sin^2\left(\frac{n\pi \log x}{\log 2}\right) dx = 1.$$

LET $y = \log x / \log 2$ SO THAT $dy = \dfrac{1}{x \log 2}\,dx$

$$\longrightarrow \quad c^2 \log 2 \int_0^1 \sin^2(n\pi y)\,dy = 1 \rightarrow c^2(\log 2)/2 = 1.$$

THIS YIELDS THAT $C = (2/\log 2)^{1/2}$.

FINALLY, WE OBTAIN THAT

$$\phi_n(x) = \left(\frac{2}{\log 2}\right)^{1/2} x^{-1/2} \sin\left(\frac{n\pi \log x}{\log 2}\right) \qquad n = 1, 2, 3, \ldots$$

$$\lambda_n = \frac{1}{4} + \frac{n^2\pi^2}{(\log 2)^2}.$$

NOW IF WE EXPAND $f(x)$ IN TERMS OF THIS SERIES

WE OBTAIN $\quad f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \qquad c_n = \int_1^2 f(x)\,\phi_n(x)\,dx.$

<u>EXAMPLE</u> SOLVE THE HEAT CONDUCTION PROBLEM

$$\begin{cases} u_t = D(x^2 u_x)_x - u & 1 \le x \le 2,\ t > 0 \\ u(1,t) = u(2,t) = 0, & u(x,0) = f(x) \end{cases}$$

SEPARATING VARIABLES WE OBTAIN $u(x,t) = X(x)\,T(t)$

THEN $\quad X T' = D T (x^2 X')' - X T \quad \rightarrow \frac{1}{D}\left(\frac{T'}{T} + 1\right) = \frac{(x^2 X')'}{X} = -\lambda.$

THIS LEADS TO THE EIGENVALUE PROBLEM

$$\left.\begin{aligned} &(x^2\phi')' + \lambda\phi = 0 \qquad 1 < x < 2 \\ &\phi(1) = \phi(2) = 0 \end{aligned}\right\} \qquad \phi_k(x) = \left(\frac{2}{\log 2}\right)^{1/2} x^{-1/2} \sin\left(\frac{n\pi \log x}{\log 2}\right)$$

$$\lambda_k = \frac{1}{4} + \frac{k^2\pi^2}{(\log 2)^2}, \quad k = 1, 2, \ldots$$

WE THEN OBTAIN $\quad T_k' = -(1 + D\lambda_k)\,T$

THIS YIELDS THAT $\quad T_k(t) = e^{-t} e^{-D\lambda_k t}$

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(1+D\lambda_n)t} \phi_n(x) \qquad \text{WITH } u(x,0) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

THUS $\quad f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \quad$ SO THAT $\quad c_n = \int_1^2 f(x)\,\phi_n(x)\,dx.$

SPECIAL CASES OF STURM LIOUVILLE PROBLEMS

(i) $\phi'' + x\phi' + \lambda\phi = 0$

$\phi(0) = \phi(1) = 0$

MULTIPLY BY $e^{x^2/2}$ SO THAT $(e^{x^2/2}\phi)' + \lambda e^{x^2/2}\phi = 0$

$\phi(0) = \phi(1) = 0.$

THE WEIGHT FUNCTION IS $w = e^{x^2/2}$ SO THAT $\int_0^1 \phi_n \phi_m e^{x^2/2}\,dx = 0$ IF $n \neq m$.

(ii) $\phi'' + \frac{2}{x}\phi' + \lambda\phi = 0$

$\phi(1) = \phi(2) = 0$

IN STURM-LIOUVILLE FORM $(x^2\phi')' + \phi\lambda x^2 = 0$

THE WEIGHT FUNCTION IS $w = x^2$ AND $\int_0^1 \phi_n \phi_m x^2\,dx = 0$, $n \neq m$

WE CAN SOLVE FOR ϕ BY WRITING $\phi(x) = v(x)/x$

SO THAT $v'' + \lambda v = 0$ AND $v = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$.

THIS YIELDS $\phi = \frac{1}{x}\left[A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)\right]$

IT IS MORE CONVENIENT TO WRITE

$$\phi = \frac{1}{x}\left[A\cos(\sqrt{\lambda}\,(x-1)) + B\sin(\sqrt{\lambda}\,(x-1))\right].$$

NOW $\phi(1) = 0 \rightarrow A = 0$. $\phi(2) = 0 \rightarrow \sin(\sqrt{\lambda}) = 0$

HENCE $\sqrt{\lambda} = n\pi$ OR $\lambda = n^2\pi^2$

AND $\phi_n(x) = \frac{1}{x}\sin(n\pi(x-1))$

WHEN $p(x)$ VANISHES AT ONE OF THE ENDPOINTS, OR WHEN EITHER ENDPOINT IS ∞ WE HAVE A <u>SINGULAR</u> STURM-LIOUVILLE PROBLEM.

(iii) $\phi'' + \frac{1}{x}\phi' + \lambda\phi = 0, \quad 0 < x < 1; \ \phi(0)$ FINITE, $\phi(1) = 0$.

THIS IS BESSEL'S EQUATION WITH $\phi = A J_0(\sqrt{\lambda}\, x) + B Y_0(\sqrt{\lambda}\, x)$

AND IN STURM-LIOUVILLE FORM $(x\phi')' + \lambda x \phi = 0$

SO THAT $\int_0^1 x\, \phi_n \phi_m \, dx = 0$ FOR $n \neq m$.

(iv) $\phi'' - 2x\phi' + \lambda\phi = 0 \quad -\infty < x < \infty$; HERMITE'S EQUATION

IN STURM-LIOUVILLE FORM $(e^{-x^2}\phi')' + \lambda e^{-x^2}\phi = 0$

SO THAT $\int_{-\infty}^{\infty} e^{-x^2} \phi_n \phi_m \, dx = 0$ FOR $n \neq m$.

USING FROBENIUS SERIES THERE ARE POLYNOMIAL SOLUTIONS TO THIS EQUATION WHEN $\lambda = 2n$, FOR $n = 0, 1, 2, \ldots$

THEN $\phi_n(x) = H_n(x)$ HERMITE POLYNOMIALS

$$\lambda = 2n, \quad n = 0, 1, 2, \ldots$$

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \ldots$$

(v) $\phi'' - \frac{2x}{1-x^2}\phi' + \frac{\lambda}{1-x^2}\phi = 0$ IN $-1 < x < 1$, $\phi(\pm 1)$ FINITE.

THIS IS LEGENDRE'S EQUATION AND IN SL FORM WE GET

$$[(1-x^2)\phi']' + \lambda\phi = 0 \quad \text{so} \quad \int_{-1}^{1} \phi_n(x)\phi_m(x)\, dx = 0 \quad n \neq m$$

THE SOLUTIONS ARE $\phi = A P_n(x) + B Q_n(x)$ WHEN $\lambda = n(n+1)$.

NOW $P_n(x)$ ARE LEGENDRE POLYNOMIALS OF DEGREE n.

<u>REMARKS</u> IN GENERAL FOR EIGENVALUE PROBLEMS WITH NON-SEPARATED BOUNDARY CONDITIONS SUCH AS

$$(*)\quad \begin{cases} \phi'' + \lambda\phi = 0, & 0 < x < 1 \\ \phi(0) = 0 \\ \phi'(1) = h\,\phi'(0) \end{cases}$$

$\longleftarrow$ HERE THE CONDITION AT $x=1$ DEPENDS ON THAT AT $x=0$ $\rightarrow$ NON-SEPARATED BC

WE CAN EXPECT THE POSSIBILITY OF COMPLEX EIGENVALUES. THIS IS BECAUSE THE PROOF THAT EIGENVALUES ARE REAL FAILS SINCE LAGRANGE'S IDENTITY IN EQUATION (1) ON PAGE 6 DOES NOT GIVE $\int_0^1 (u\mathcal{L}v - v\mathcal{L}u)\,dx = 0$.

AN ANALYSIS OF $(*)$ SHOWING AN INFINITE # OF COMPLEX EIGENVALUES WHEN $h>1$ IS DONE IN THE HOMEWORK.

HOWEVER, THERE IS ONE TYPE OF NON-SEPARATED BOUNDARY CONDITIONS WHICH OCCURS OFTEN AND LEADS TO REAL EIGENVALUES. CONSIDER THE CASE OF <u>PERIODIC BOUNDARY</u> CONDITIONS:

$$\phi'' + \lambda\phi = 0, \quad 0 < x < L; \qquad \phi(0) = \phi(L), \quad \phi'(0) = \phi'(L).$$

FOR THIS PROBLEM WE CALCULATE FOR $\lambda > 0$ THAT $\phi(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$.

WE PUT $\phi(0) = \phi(L) \rightarrow A = A\cos(\mu) + B\sin(\mu)$ $\qquad \mu = \sqrt{\lambda}\,L$

$\phi'(0) = \phi'(L) \rightarrow B = -A\sin(\mu) + B\cos(\mu)$

THIS GIVES $\underbrace{\begin{pmatrix} 1-\cos\mu & -\sin\mu \\ \sin\mu & 1-\cos\mu \end{pmatrix}}_{A} \begin{pmatrix} A \\ B \end{pmatrix} = \underline{0}$ WHICH HAS A NON-TRIVIAL SOLUTION WHEN $\det A = 0$.

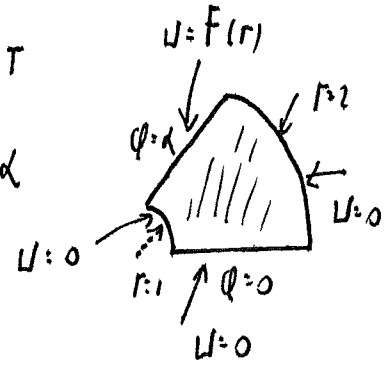
THUS THE EIGENVALUES CORRESPOND TO $\det A = 0 \rightarrow (1-\cos\mu)^2 + \sin^2\mu = 2 - 2\cos\mu = 0$.

THUS, $\cos\mu = 1 \rightarrow \mu = \sqrt{\lambda}\,L = 2n\pi$, $n = 0, 1, 2, \ldots$ THE EIGENVALUES ARE

$\lambda_n = (2n\pi/L)^2$ FOR $n = 1, 2, \ldots$ AND $\phi_n = A\cos\left(\frac{2n\pi x}{L}\right) + B\sin\left(\frac{2n\pi x}{L}\right)$.

EXAMPLE (EULER'S EQUATION)

SOLVE LAPLACE'S EQUATION IN A WEDGE WITH CUT-OUT



$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} = 0 \quad \text{IN} \quad 1 \le r \le 2, \quad 0 \le \varphi \le \alpha$$

$$u(1,\varphi) = u(2,\varphi) = 0$$

$$u(r,\alpha) = F(r), \quad u(r,0) = 0$$

SOLUTION WE LOOK FOR AN EIGENFUNCTION EXPANSION IN RADIAL DIRECTION. SEPARATE VARIABLES:

$$u = R(r)\,\Phi(\varphi)$$

then
$$\left(R'' + \frac{1}{r} R'\right)\Phi + \frac{1}{r^2} R\,\Phi'' = 0.$$

so
$$\frac{r^2 R'' + r R'}{R} = -\frac{\Phi''}{\Phi} = -\lambda. \longrightarrow \quad \Phi'' - \lambda\Phi = 0, \quad \Phi(0) = 0.$$

so $(+)\Big\}$ $r^2 R'' + r R' + \lambda R = 0, \qquad R(1) = R(2) = 0$

THIS GIVES IN SL FORM:

$$\Big\} \; r R'' + R' + \frac{\lambda}{r} R = (r R')' + \frac{\lambda}{r} R = 0.$$

THUS
$$(*)\Bigg\} \begin{array}{l} (rR')' + \dfrac{\lambda}{r} R = 0 \quad \text{IN} \quad 1 \le r \le 2 \\ R(1) = R(2) = 0 \end{array}$$

- FROM $(*)$ WE IDENTIFY THE weight FUNCTION AS $W = 1/r$.
- ALSO FROM $(*)$ WE CLAIM $\lambda > 0$.

PROOF MULTIPLY BY R: $\quad R(rR')' + \lambda \dfrac{R^2}{r} = 0$

AND INTEGRATE
$$\int_1^2 R(rR')'\,dr = -\lambda \int_1^2 \frac{R^2}{r}\,dr.$$

NOW INTEGRATE BY PARTS
$$R(rR')\Big|_1^2 - \int_1^2 r(R')^2\,dr = -\lambda \int_1^2 \frac{R^2}{r}\,dr.$$

BUT $R(1) = R(2) = 0$. HENCE, BOUNDARY TERM VANISHES, AND

$$+\underbrace{\int_1^2 r(R')^2\,dr}_{\geq 0} = +\lambda \underbrace{\int_1^2 \frac{R^2}{r}\,dr}_{\geq 0} \longrightarrow \lambda \geq 0.$$

BUT $\lambda = 0$ IS NOT AN eigenvalue SINCE IF WE SET $\lambda = 0$ we get

$(rR')' = 0 \rightarrow rR' = c$ so $R = c\log r + b$.

BUT $R(1) = R(2) = 0 \rightarrow c = b = 0 \Rightarrow R \equiv 0$ NOT AN eigenfunction.

THUS NEED ONLY CONSIDER $\lambda > 0$ IN EULER'S EQUATION (+). WE WRITE $R = r^B$ IN (+). WE get $B(B-1) + B + \lambda = 0 \rightarrow B^2 = -\lambda$.

BUT $\lambda > 0$, so $B = \pm i\sqrt{\lambda}$.

WITHOUT LOSS OF generality take + SIGN. $R = r^{i\sqrt{\lambda}} = e^{i\sqrt{\lambda}\log r}$

$$R = e^{i\sqrt{\lambda}\log r} = \cos(\sqrt{\lambda}\log r) + i\sin(\sqrt{\lambda}\log r) \quad \text{IS}$$

A COMPLEX-VALUED SOLUTION. SINCE $\mathrm{IM}[R]$, $\mathrm{REM}[R]$ ARE BOTH SEPARATELY SOLUTIONS, THEN GENERAL SOLUTION IS

$$R(r) = c_1\cos(\sqrt{\lambda}\log r) + c_2\sin(\sqrt{\lambda}\log r).$$

BUT $R(1) = 0 \longrightarrow c_1 = 0$

$R(2) = 0 \longrightarrow c_2\sin(\sqrt{\lambda}\log 2) = 0 \rightarrow \sqrt{\lambda}\log 2 = n\pi$

so $\lambda_n = n^2\pi^2/(\log 2)^2$, $n = 1, 2, 3, \ldots$

$$R_n(r) = \sin\left(\frac{n\pi\log r}{\log 2}\right) \quad n = 1, 2, \ldots \quad \text{AND} \quad \int_1^2 \frac{1}{r} R_n R_m\,dr = 0 \quad n \neq m.$$

NOW RECALL $\Phi'' - \lambda \Phi = 0$.

$$\Phi(0) = 0$$

SO $$\Phi_n(\varphi) = A_n \sinh[\lambda_n \varphi] + B_n \cosh[\lambda_n \varphi].$$

BUT $\Phi_n(0) = 0 \rightarrow B_n = 0$.

THIS GIVES BY SUPERPOSITION

$$(1) \left\{ \quad u(r,\varphi) = \sum_{n=1}^{\infty} A_n \sinh[\lambda_n \varphi] R_n(r) \right.$$

NOW AS A FINAL-STEP SET $u(r,\alpha) = F(r)$.

THEN $$F(r) = \sum_{n=1}^{\infty} A_n \sinh[\lambda_n \alpha] R_n(r).$$

MULTIPLY BY THE weight $W = 1/r$ AND $R_m(r)$ AND INTEGRATE.

$$\int_1^2 \frac{1}{r} F(r) R_m(r) = \sum_{n=1}^{\infty} A_n \sinh[\lambda_n \alpha] \int_1^2 \frac{1}{r} R_n R_m \, dr.$$

USING ORTHOGONALITY

$$A_m \sinh[\lambda_m \alpha] = \frac{\int_1^2 \frac{1}{r} F(r) R_m(r)}{\int_1^2 \frac{1}{r} (R_m(r))^2 \, dr} = \frac{2}{\log 2} \int_1^2 \frac{F(r) R_m(r)}{r} dr$$

WHICH DETERMINES A_m IN SOLUTION (1).

<u>NOTE</u>: $$\int_1^2 \frac{1}{r} R_m^2(r) \, dr = \int_1^2 \frac{1}{r} \sin^2\left(\frac{m\pi \log r}{\log 2}\right) dr \qquad \text{let } t = \log r, \quad dt = 1/r \, dr$$

$$= \int_0^{\log 2} \sin^2\left(\frac{m\pi t}{\log 2}\right) dt = \frac{1}{2} \log 2.$$

INHOMOGENEOUS PROBLEMS AND EIGENFUNCTION EXPANSIONS

THERE ARE <u>TWO SOURCES</u> OF INHOMOGENEOUS TERMS:

(i) <u>IN THE EQUATION</u>, i.e.

$$u_t = u_{rr} + \frac{2}{r} u_r + G(r,t) \quad , \quad 0 \le r \le a, \quad t > 0$$

$$u(a,t) = 0, \quad u(0,t) \text{ BOUNDED}$$

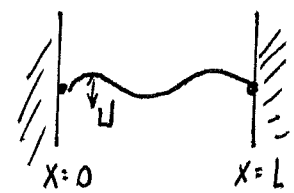
WHERE $G(r,t)$ MEASURES INTERNAL HEATING IN A SPHERE OF RADIUS a.

ANOTHER EXAMPLE IS

$$u_{tt} = c^2 u_{xx} + F(x,t) \qquad 0 < x < L, \quad t > 0$$

$$u = 0 \text{ AT } x = 0, L$$

$$u(x,0) = f(x)$$



WHERE $F(x,t)$ MEASURES EXTERNAL FORCE ON A STRING WHOSE LINEARIZED DEFLECTION IN VERTICAL DIRECTION IS u.

FOR A BIRD JUMPING ON A WIRE WE CAN MODEL

$F(x,t) = \delta(x-x_0)\sin(wt)$, WHERE $\delta(x-x_0)$ IS DIRAC DELTA FUNCTION.

FOR THESE PROBLEMS WHERE INHOMOGENEOUS TERMS APPEAR IN THE EQUATION WE WILL SHOW LATER THAT

$$u(r,t) = \sum_{n=1}^{\infty} b_n(t)\, \phi_n(r)$$

WHERE $\phi_n(r)$ ARE EIGENFUNCTIONS OF SL PROBLEM AND WHERE $b_n(t)$ WILL SATISFY ODE'S TO BE DERIVED.

(ii) <u>INHOMOGENEOUS TERMS IN THE BOUNDARY CONDITIONS</u>

CONSIDER
$$u_t = u_{xx} \quad , 0 < x < L \, , \; t \geq 0$$
$$u(0,t) = T_0 \, , \quad u_x(L,t) = -h\,(u(L,t) - T_1) \; ; \quad u(x,0) = u_0(x)$$

WHERE T_0, T_1 (POSSIBLY DEPENDENT ON TIME t) ARE THE INHOMOGENEOUS TERMS. WE CAN'T DO SEPARATION OF VARIABLES IMMEDIATELY SINCE IF WE PUT $u(x,t) = \phi(x)\,T(t) \longrightarrow u(0,t) = T_0$ GIVES $T_0 = \phi(0)\,T(t)$ WITH $\phi(0) = 0$ WHICH IS A CONTRADICTION. WE MUST MAKE A TRANSFORMATION TO GET HOMOGENEOUS BOUNDARY CONDITIONS.

PUT $\quad u(x,t) = u_s + v(x,t)$

WHERE v IS TO HAVE HOMOGENEOUS BC AND WHERE
$$u_{s_{xx}} = 0 \quad ; \quad u_s = T_0 \text{ AT } x = 0 \, , \quad u_{s_x} = -h\,(u_s - T_1) \text{ AT } x = L.$$

THE SOLUTION IS $\quad u_s = Ax + B \quad$ WHERE $\quad T_0 = B \quad$ AND $\quad A = -h\,[AL + T_0 - T_1]$

THUS $\quad A = \dfrac{-h\,(T_0 - T_1)}{1 + hL} \quad \Longrightarrow \quad u_s = \dfrac{-h\,(T_0 - T_1)}{1 + hL}\,x + T_0 .$

NOW OBTAIN EQUATION FOR v:
$$\partial_t u_s + v_t = u_{s_{xx}} + v_{xx} \qquad \text{BUT } u_{s_{xx}} = 0 \text{ SO}$$

$$(+) \quad \begin{cases} v_t = v_{xx} + G(x,t) \, , \quad 0 < x < L, \quad t > 0 \qquad G(x,t) \equiv -\partial_t u_s \\ v(0,t) = 0 \, , \quad v_x(L,t) = -h\,v(L,t) \\ v(x,0) = u(x,0) - u_s\big|_{t=0} \end{cases}$$

THE TRANSFORMED PROBLEM (+) CAN BE SOLVED BY SEPARATION OF VARIABLES BY LOOKING FOR A SOLUTION IN THE FORM
$$v(x,t) = \sum_{n=1}^{\infty} b_n(t)\,\phi_n(x)$$
WHERE $\phi_n(x)$ ARE EIGENFUNCTIONS OF SL.

KEY POINT IF THERE ARE INHOMOGENEOUS TERMS IN THE BOUNDARY CONDITIONS YOU MUST FIRST MAKE A TRANSFORMATION TO GET A PROBLEM WITH HOMOGENEOUS BOUNDARY CONDITIONS, AT THE EXPENSE OF INSERTING AN INHOMOGENEOUS TERM INTO THE EQUATION.

REMARK 1 IN PREVIOUS EXAMPLE IF T_0, T_1 ARE CONSTANTS INDEPENDENT OF TIME, THEN $u_s(x) = -\frac{h(T_0 - T_1)}{1+hL}x + T_0$ IS THE STEADY-STATE SOLUTION AND $v(x,t)$ SATISFIES

$$(++)\begin{cases} v_t = v_{xx} \\ v(0,t) = 0, \quad v_x(L,t) = -h\,v(L,t) \\ v(x,0) = u(x,0) - u_s(x) \end{cases}$$

AS A SECOND EXAMPLE CONSIDER

$$u_t = u_{rr} + \frac{2}{r}u_r \qquad 0 \le r \le a, \quad t > 0$$

$$u(a,t) = \sin(\omega t) \; ; \quad u(r,0) \text{ NICE}$$

$$u(r,0) = f(r).$$

TO ELIMINATE INHOMOGENEOUS TERM IN BC LET $u(r,t) = u_s + v(r,t)$ WHERE

$$u_{srr} + \frac{2}{r}u_{sr} = 0, \quad 0 \le r \le a,$$

$$u_s = \sin(\omega t) \text{ AT } r = a, \quad u_s \text{ NICE as } r \to 0.$$

THE SOLUTION IS $u_s = A + B/r$ WITH $B = 0$ AND $A = \sin(\omega t)$. IF WE PUT $u = u_s + v$ WE GET

$$\begin{cases} v_t = v_{rr} + \frac{2}{r}v_r - \omega\cos\omega t \\ v(a,t) = 0, \quad v(0,t) \text{ NICE} \\ v(r,0) = f(r) - u_s|_{t=0} = f(r). \end{cases}$$

<u>EXAMPLE</u> CONSIDER HEAT FLOW IN A SPHERE WITH PERIODIC TEMPERATURE VARIATIONS ON ITS BOUNDARY MODELLED BY

$$u_t = D\left(u_{rr} + \frac{2}{r}u_r\right) \quad \text{IN} \quad 0 \le r \le a,\ t \ge 0$$

WITH (BC) $\quad u(a,t) = \sin(\omega t)$; u, u_r BOUNDED AS $r \to 0$

AND (IC) $\quad u(r,0) = 0$.

WE FIRST MUST GET HOMOGENEOUS BC. DEFINE U BY

$$U_{rr} + \frac{2}{r}U_r = 0 \ ; \qquad U = \sin(\omega t) \text{ ON } r = a, \quad U, U' \text{ BOUNDED AS } r \to 0.$$

WE GET $U = d_0 + d_1/r$. THEN U NICE AS $r \to 0 \Rightarrow d_1 = 0$ AND CONDITION AT $r = a$ GIVES $d_0 = \sin(\omega t)$. WE GET

$$U = \sin(\omega t).$$

WE WRITE $\quad u(r,t) = U + v(r,t)$ AND SUBSTITUTE:

$$U_t + v_t = D\left(U_{rr} + \frac{2}{r}U_r\right) + D\left(v_{rr} + \frac{2}{r}v_r\right)$$

$$v(a,t) = \sin(\omega t) - U\big|_{r=a}$$

$$v(r,0) = u(r,0) - U\big|_{t=0}$$

WE GET BY USING $U = \sin(\omega t)$ THAT

$$v_t = D\left(v_{rr} + \frac{2}{r}v_r\right) - \omega\cos(\omega t) \quad \text{IN} \quad 0 \le r \le a,\ t \ge 0$$

(BC) $\quad v(a,t) = 0$, $\quad v, v_r$ BOUNDED AS $r \to 0$

(IC) $\quad v(r,0) = 0$.

THE UNDERLYING SL PROBLEM IS

$$\Phi'' + \frac{2}{r}\Phi' + \lambda\Phi = 0$$

$$\Phi(a) = 0 \ ; \quad \Phi, \Phi' \text{ BOUNDED AS } r \to 0$$

$$\Longrightarrow \quad \Phi_n = \sin\left(\frac{n\pi r}{a}\right)\Big/ r$$

$$\lambda_n = n^2\pi^2/a^2, \quad n = 1, 2, \ldots$$

$$\int_0^a r^2 \Phi_n \Phi_m \, dr = 0 \quad \text{IF } n \ne m$$

NOW EXPAND $V(r,t) = \sum_{n=1}^{\infty} b_n(t)\,\Phi_n(r)$.

SUBSTITUTE TO GET $\sum_{n=1}^{\infty} b_n'\,\Phi_n = D\sum_{n=1}^{\infty} b_n\left(\Phi_n'' + \frac{2}{r}\Phi_n'\right) - w\cos(wt)$.

USING EIGENVALUE RELATION $\Phi_n'' + \frac{2}{r}\Phi_n' = -\lambda_n\Phi_n$ YIELDS

$$\sum_{n=1}^{\infty}\left(b_n' + D\lambda_n b_n\right)\Phi_n = -w\cos(wt). \qquad (1)$$

NOW EXPAND $1 = \sum_{n=1}^{\infty} f_n\,\Phi_n(r)$ SO

$$f_n = \frac{\int_0^a r^2\,\Phi_n\,dr}{\int_0^a r^2\,\Phi_n^2\,dr} = \frac{\int_0^a r\sin\left(\frac{n\pi r}{a}\right)dr}{\int_0^a \sin^2\left(\frac{n\pi r}{a}\right)dr} \qquad (2)$$

WE THEN WRITE (1) AS

$$\sum_{n=1}^{\infty}\left(b_n' + D\lambda_n b_n\right)\Phi_n = \sum_{n=1}^{\infty} f_n(-w\cos wt)\,\Phi_n$$

WE CONCLUDE THAT BY ORTHOGONALITY, AND THE FACT THAT $V(r,0)=0 \rightarrow b_n(0)=0$ THAT

$$\left.\begin{aligned} b_n' + D\lambda_n b_n &= C_n\cos(wt) \qquad n=1,2,3,\ldots \\ b_n(0) &= 0 \end{aligned}\right\} \qquad (4)$$

WHERE $C_n = -w f_n$ AND f_n DEFINED BY (2).

AFTER SOLVING FOR $b_n(t)$ AND RECALLING

$$u(r,t) = \sin(wt) + \sum_{n=1}^{\infty} b_n(t)\,\Phi_n(r) \qquad (5)$$

WE HAVE $u(r,t)$. LET'S DERIVE AN EXPRESSION FOR $u(0,t)$, i.e. THE TEMPERATURE AT CENTER OF SPHERE, VALID FOR LARGE t. SINCE

$$\lim_{r\to 0}\Phi_n(r) = \lim_{r\to 0}\frac{\sin(n\pi r/a)}{r} = \frac{n\pi}{a}$$ WE CONCLUDE THAT

$$u(0,t) = \sin(wt) + \sum_{n=1}^{\infty}\left(\frac{n\pi}{a}\right)b_n(t) \qquad (6).$$

NOW WE CALCULATE C_n AND THEN FIND $b_n(t)$.

FROM (2) WE HAVE $f_n = \frac{2}{a}\int_0^r \underbrace{r}_{u} \underbrace{\sin\left(\frac{n\pi r}{a}\right) dr}_{dv} = \frac{2}{a}\left[-r\frac{a}{n\pi}\cos\left(\frac{n\pi r}{a}\right)\right]_0^a + \frac{a}{n\pi}\int_0^a \cos\left(\frac{n\pi r}{a}\right)$

SO $$f_n = \frac{2}{a}\frac{(-a^2)}{n\pi}(-1)^n = \frac{2a}{n\pi}(-1)^{n+1}$$

THUS $$C_n = \frac{2aw(-1)^n}{n\pi}$$ TO BE USED IN (4).

NOW CALCULATE THE SOLUTION b_n:

<u>METHOD</u> $$\left(e^{D\lambda_n t} b_n\right)' = C_n e^{D\lambda_n t}\cos(wt)$$

SO $$e^{D\lambda_n t} b_n = C_n \int_0^t e^{D\lambda_n \tau}\cos(w\tau)\,d\tau$$ SATISFIES $b_n(0) = 0$.

NOW $$\int_0^t e^{D\lambda_n \tau}\cos(w\tau)\,d\tau = \mathrm{RE}\left[\int_0^t e^{(D\lambda_n + iw)\tau}\right] = \mathrm{RE}\left[\frac{1}{D\lambda_n + iw}\left[e^{(D\lambda_n + iw)t} - 1\right]\right]$$

$$= \mathrm{RE}\left[\frac{(D\lambda_n - iw)}{(D\lambda_n)^2 + w^2}\left[e^{D\lambda_n t}(\cos wt + i\sin wt) - 1\right]\right]$$

$$= \frac{1}{(D\lambda_n)^2 + w^2}\left[D\lambda_n\left(e^{D\lambda_n t}\cos(wt) - 1\right) + w e^{D\lambda_n t}\sin(wt)\right]$$

THUS $$e^{D\lambda_n t} b_n = \frac{C_n}{(D\lambda_n)^2 + w^2}\left[D\lambda_n\left(e^{D\lambda_n t}\cos(wt) - 1\right) + w e^{D\lambda_n t}\sin(wt)\right]$$

SO $$b_n(t) = \frac{C_n}{(D\lambda_n)^2 + w^2}\left[D\lambda_n\cos(wt) + w\sin(wt) - D\lambda_n e^{-\lambda_n D t}\right]$$

WITH $C_n = \frac{2aw(-1)^n}{n\pi}$. NOW IN (6) WE NEED $\frac{n\pi}{a} b_n$ SO THAT

$$\frac{n\pi}{a} b_n = \frac{2w(-1)^n}{(D\lambda_n)^2 + w^2}\left[D\lambda_n\cos(wt) + w\sin(wt) - D\lambda_n e^{-\lambda_n D t}\right] \qquad (7)$$

WE NOW SUBSTITUTE INTO (6) TO GET, UPON LETTING $t \to \infty$ TO KILL EXPONENTIAL IN (7):

$$u(0,t) \cong \sin(\omega t) + \sum_{n=1}^{\infty} \frac{2\omega(-1)^n}{(D\lambda_n)^2+\omega^2}\left(D\lambda_n \cos(\omega t) + \omega \sin(\omega t)\right), \quad \text{FOR} \quad t \gg 1.$$

NOW WRITE $\quad D\lambda_n \cos(\omega t) + \omega \sin(\omega t) = R_n \sin(\omega t - \phi_n)$

SO $\quad R_n \cos\phi_n = \omega_n, \quad R_n \sin\phi_n = -D\lambda_n \rightarrow R_n = \sqrt{(D\lambda_n)^2+\omega^2}$

$$\tan\phi_n = -D\lambda_n/\omega .$$

THEN WE HAVE COMPACTLY THAT

$$u(0,t) \cong \sin(\omega t) + \sum_{n=1}^{\infty} \frac{2\omega(-1)^n}{\sqrt{(D\lambda_n)^2+\omega^2}} \sin(\omega t - \phi_n) \quad \text{FOR} \quad t \gg 1.$$

NOW USE $\quad \dfrac{D\lambda_n}{\omega} = \dfrac{D n^2 \pi^2/a^2}{\omega} \equiv \alpha n^2 \quad$ WHERE $\quad \alpha \equiv \dfrac{D\pi^2}{a^2\omega}$.

OUR EXPLICIT FORMULA BECOMES

$$u(0,t) \cong \sin(\omega t) + \sum_{n=1}^{\infty} \frac{2(-1)^n}{\sqrt{\alpha^2 n^4 + 1}} \sin(\omega t - \phi_n) \quad \text{FOR} \quad t \gg 1$$

$$\text{WHERE} \quad \tan\phi_n = -\alpha n^2 \quad \text{AND} \quad \alpha = \frac{D\pi^2}{a^2\omega}.$$

NOTICE THAT α IS DIMENSIONLESS SINCE $[D] = \text{length}^2/\text{TIME}$, $[a^2] = \text{length}^2$ AND $[\omega] = 1/\text{TIME}$. IF α IS LARGE, I.E. FAST DIFFUSION, THEN THE INFINITE SUM TERM IS <u>SMALL</u> AND SO $u(0,t)$ IS CLOSE TO $u(a,t)$ AS WE WOULD EXPECT.

<u>EXAMPLE</u> FIND THE SOLUTION TO THE FORCED WAVE EQUATION

$$u_{tt} = c^2 u_{xx} + F(x,t) \quad , \quad 0 < x < L, \ t > 0$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = g(x), \quad u_t(x,0) = 0$$

WHERE $F(x,t) = \delta(x - x_0) \sin\left(\frac{N\pi c t}{L}\right)$ N a positive integer.

THIS CORRESPONDS TO PLUCKING THE STRING LOCALLY AT SOME $x = x_0$ WITH $0 < x_0 < L$.

<u>REMARK</u> RECALL THAT $\int_0^1 \delta(x - x_0) g(x)\, dx = g(x_0)$ WHEN $0 < x_0 < 1$.

ALSO $\delta(x - x_0) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-x_0)^2/2\sigma^2}$ IN generalized FUNCTION SENSE:

$$\delta(x - x_0) = \begin{cases} 0 & \text{IF } x \neq x_0 \\ \infty & \text{IF } x = x_0 \end{cases} \qquad 1 = \int_a^b \delta(x - x_0)\, dx$$

FOR $a < x_0 < b$.

FOR THE HOMOGENEOUS PROBLEM THE EIGENFUNCTIONS ARE $\sin(n\pi x/L)$ AND THE NORMALIZED EIGENFUNCTIONS ARE

$$\phi_n(x) = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n\pi x}{L}\right) \quad \text{WITH} \quad \int_0^L (\phi_n(x))^2\, dx = 1$$

WE LOOK FOR A SOLUTION OF THE FORM

$$(*) \qquad u(x,t) = \sum_{n=1}^{\infty} b_n(t)\, \phi_n(x).$$

THEN

$$u(x,0) = g(x) = \sum_{n=1}^{\infty} b_n(0)\, \phi_n(x) \longrightarrow b_n(0) = \int_0^L g(x)\, \phi_n(x)\, dx$$

$$u_t(x,0) = 0 = \sum_{n=1}^{\infty} b_n'(0)\, \phi_n(x) \longrightarrow b_n'(0) = 0$$

WE THEN SUBSTITUTE $(*)$ INTO THE PDE TO OBTAIN

$$\sum_{n=1}^{\infty} \left(b_n'' + \omega_n^2 b_n\right) \phi_n(x) = \sum_{n=1}^{\infty} d_n(t)\, \phi_n(x),$$

WITH $\omega_n = c n\pi/L$ AND $d_n(t) = \int_0^L \phi_n(x)\, F(x,t)\, dx = \int_0^L \phi_n(x)\, F(x,t)\, dx.$

NOW WE CALCULATE

$$d_n(t) = \int_0^L \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n\pi x}{L}\right) \delta(x-x_0) \sin\left(\frac{N\pi c t}{L}\right) dx$$

THIS YIELDS $\quad d_n(t) = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n\pi x_0}{L}\right) \sin(\omega_N t), \quad \omega_N = N\pi c/L.$

BY ORTHOGONALITY $\quad u(x,0) = g(x) = \sum_{n=1}^{\infty} b_n(0)\, \phi_n(x).$

THIS YIELDS THAT $\quad b_n(0) = \int_0^L g(x)\, \phi_n(x)\, dx$

THE PROBLEM FOR $b_n(t)$ BECOMES:

$$\begin{cases} b_n'' + \omega_n^2 b_n = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n\pi x_0}{L}\right) \sin(\omega_N t), \quad \omega_n = n\pi c/L \\ b_n(0) = \int_0^L g(x)\, \phi_n(x)\, dx, \quad b_n'(0) = 0 \end{cases}$$

<u>CASE 1</u> $(n \neq N)$ NON-RESONANT CASE

THE PARTICULAR SOLUTION HAS THE FORM

$b_n(t) = A_n \sin(\omega_N t)$ SO THAT $A_n(\omega_n^2 - \omega_N^2) = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n\pi x_0}{L}\right).$

THIS YIELDS $\quad A_n = \left(\frac{2}{L}\right)^{1/2} \frac{1}{\omega_n^2 - \omega_N^2} \sin\left(\frac{n\pi x_0}{L}\right)$

THE GENERAL SOLUTION IS

$$b_n(t) = \alpha_n \sin(\omega_n t) + B_n \cos(\omega_n t) + A_n \sin(\omega_N t)$$

BUT $b_n'(0) = 0 \rightarrow \alpha_n \omega_n + A_n \omega_N = 0 \longrightarrow \alpha_n = -\frac{\omega_N}{\omega_n} A_n.$

$b_n(0) = B_n.$

THIS YIELDS THAT FOR $n \neq N$.

$$b_n(t) = A_n\left[\sin(w_N t) - \frac{w_N}{w_n}\sin(w_n t)\right] + b_n(0)\cos(w_n t).$$

<u>CASE 2</u> $(n = N)$ RESONANT CASE.

WE WRITE THE PARTICULAR SOLUTION AS $b_{Np} = \mathrm{IM}(\tilde{b}_N)$.

THIS YIELDS $$\tilde{b}_N'' + w_N^2 \tilde{b}_N = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{N\pi x_0}{L}\right) e^{i w_N t}.$$

WE SUBSTITUTE $$\tilde{b}_N = B_N t e^{i w_N t}$$

TO OBTAIN THAT $$B_N = \frac{-i}{2w_N}\left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{N\pi x_0}{L}\right)$$

THIS YIELDS THAT $$b_{Np}(t) = \mathrm{IM}(\tilde{b}_N) = -\frac{1}{2w_N}\left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{N\pi x_0}{L}\right) t\cos(w_N t)$$

FINALLY WE WRITE

$$b_N(t) = \alpha_N \sin(w_N t) + B_N \cos(w_N t) - \frac{t}{2w_N}\left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{N\pi x_0}{L}\right)\cos(w_N t)$$

$$b_N'(0) = 0 \longrightarrow \alpha_N w_N - \frac{1}{2w_N}\left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{N\pi x_0}{L}\right) = 0$$

$$\alpha_N = \frac{1}{2w_N^2}\left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{N\pi x_0}{L}\right)$$

THIS YIELDS THAT

$$b_N(t) = b_N(0)\cos(w_N t) + \frac{1}{2w_N^2}\left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{N\pi x_0}{L}\right)\sin(w_N t) - \frac{t}{2w_N}\left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{N\pi x_0}{L}\right)\cos(w_N t)$$

THE FINAL SOLUTION IS

$$u(x,t) = b_N(t)\sin\left(\frac{N\pi x}{L}\right) + \sum_{n=1,\, n\neq N}^{\infty} b_n(t)\left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n\pi x}{L}\right)$$

INHOMOGENEOUS PROBLEMS: EIGENFUNCTION EXPANSIONS

SUPPOSE THAT WE HAVE A HEAT EQUATION ON $0 \le x \le 1$, $t > 0$

$$w(x)\, u_t = (p(x)\, u_x)_x - q(x)\, u + G(x,t) \qquad p(x) > 0 \text{ ON } 0 < x < 1$$

$$w(x) > 0 \text{ ON } 0 < x < 1$$

$$u_x(0,t) = h_1\, u(0,t) \qquad h_1 \geq 0,\ h_2 \geq 0$$

$$u_x(1,t) = -h_2\, u(1,t)$$

$$u(x,0) = f(x)$$

WE FIRST CONSIDER THE EIGENVALUE PROBLEM FOR THE HOMOGENEOUS PROBLEM

$$(p(x)\phi')' - q(x)\phi + \lambda w(x)\phi = 0, \quad 0 < x < 1$$

$$\phi'(0) - h_1\phi(0) = 0, \quad \phi'(1) + h_2\phi(1) = 0$$

THE NORMALIZED EIGENFUNCTIONS ARE $\phi_n(x)$ WITH EIGENVALUES λ_n FOR $n = 1, 2, 3, \ldots$. THE NORMALIZATION CONDITION IS $\int_0^1 w(x)(\phi_n(x))^2\, dx = 1$.

WE LOOK FOR A SOLUTION IN THE FORM

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\, \phi_n(x)$$

WHERE $b_n(t)$ FOR $n = 1, 2, 3, \ldots$ ARE TO BE FOUND.

WE SUBSTITUTE INTO THE PDE TO OBTAIN

$$\sum_{n=1}^{\infty} w(x)\, b_n'(t)\, \phi_n(x) = \sum_{n=1}^{\infty} b_n(t)\, \mathcal{L}\phi_n + \frac{G(x,t)}{w(x)}\, w(x)$$

WHERE $\mathcal{L}\phi_n = (p\phi_n')' - q\phi_n = -\lambda_n w \phi_n$. NOW EXPAND

$$\frac{G(x,t)}{w(x)} = \sum_{n=1}^{\infty} c_n \phi_n(x) \qquad c_n = \int_0^1 G(x,t)\, \phi_n(x)\, dx.$$

THIS YIELDS THAT

$$\sum_{n=1}^{\infty} w(x)\, b_n' \phi_n = \sum_{n=1}^{\infty} -\lambda_n b_n w \phi_n + \sum_{n=1}^{\infty} c_n \phi_n w$$

EQUIVALENTLY, WE OBTAIN THAT

$$\sum_{n=1}^{\infty} \left(b_n' + \lambda_n b_n - c_n \right) \phi_n w = 0$$

BY ORTHOGONALITY OF THE EIGENFUNCTIONS ϕ_n, $n = 1, 2, \ldots$

WE OBTAIN THAT

$$(*) \quad b_n' + \lambda_n b_n = c_n .$$

NOW $\quad u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n(0)\, \phi_n(x).$

THIS YIELDS THAT $\quad b_n(0) = \int_0^1 f(x)\, \phi_n(x)\, w(x)\, dx$

BY ORTHOGONALITY AND $\int_0^1 w \phi_n^2\, dx = 1.$

WE SOLVE $(*)$ BY WRITING

$$\left(b_n e^{\lambda_n t} \right)' = c_n(t) e^{\lambda_n t}$$

WE OBTAIN $\quad b_n e^{\lambda_n t} = b_n(0) + \int_0^t e^{\lambda_n \tau} c_n(\tau)\, d\tau.$

THIS YIELDS THAT $\quad b_n(t) = b_n(0) e^{-\lambda_n t} + e^{-\lambda_n t} \int_0^t e^{\lambda_n \tau} c_n(\tau)\, d\tau.$

THE EIGENFUNCTION FUNCTION EXPANSION

SOLUTION IS $\quad u(x,t) = \sum_{n=1}^{\infty} b_n(t)\, \phi_n(x)$

WITH $b_n(t)$ AS GIVEN ABOVE.