MATH 345: LAB 5, LIMIT CYCLES AND A HOPF BIFURCATION

Purpose: The purpose of this lab is to use **xppaut** to explore limit cycle behavior in a realistic predator-prey model and how such a limit cycle is initiated and is annihilated as a parameter in the differential equation is varied. The birth of a limit cycle as a parameter is varied, which results from the Hopf-Bifurcation theorem, is also examined for a simplistic model problem.

2. THE SPECIFIC LAB PROJECT

1. **The Model Problem:** The following ODE system gives rise to the birth of a limit cycle as the parameter κ is varied across some critical value:

$$x' = \kappa x + 5y - x(x^2 + y^2)\,,$$
$$y' = -5x + \kappa y - y(x^2 + y^2)\,,$$

The system is contained in the file on my website *hopf.ode*. Please copy and paste it into your directory.

   i) Plot the phase plane on the interval $|x| < 1$, $|y| < 1$ for each of the four values $\kappa = -.20$, $\kappa = 0.0$, $\kappa = .20$, $\kappa = .50$.

   ii) Explain the qualitative features in these figures analytically by first converting the ODE system to polar coordinates. What is the critical value of κ for which a limit cycle is initiated? The birth of the limit cycle is a result of the Hopf-Bifurcation theorem to be discussed in class.

2. **Realistic Predator-Prey System:** The following ODE system models predator-prey dynamics under the assumption that the predator's appetite may become saturated by a glut of prey and the prey may become reduced due to overcrowding (i.e. a saturation effect).

$$x' = -.5x + xy/(.3 + \kappa y)\,,$$
$$y' = y - y^2 - xy/(.3 + \kappa y)\,,$$

Here x is the predator and y is the prey. The system is contained in the file on my website *lotreal.ode*. Please copy and paste it into your directory.

   i) Plot the phase plane on the interval $0 < x < 1$, $0 < y < 1.2$ for each of the three values $\kappa = 0.0$, $\kappa = .90$, $\kappa = 1.35$. For which of these values does a limit cycle occur?

   ii) See if you can determine numerically the minimum value of κ for which a limit cycle will exist.

   iii) Use xppaut to plot the branch of equilibrium and periodic solutions as κ is varied. What should happen is that a branch of periodic solutions bifurcates off of the equilibrium solution as κ is varied.

iv) Find the equilibrium point with both x and y non-zero and try to classify its stability and type for various values of κ. Does this relate to the result in ii)?

v) Can you apply the Poincare-Bendixson theorem to establish the existence of limit cycles for any values of κ? (i.e. can you construct a trapping region). You may get some help here by plotting the direction field using xppaut and then verifying analytically that a trapping region exists.

Recall:

Calling the Program: To invoke the program:

1. In your main directory, type *xppaut* and hit *return.* (this calls the executable)
2. You will then get a window that appears where you can click on the file harvest.ode that you wish to load.
3. At this stage XPPAUT opens a big window with MANY options should appear. We will examine only a few of the large number of options in this lab.

Also Recall:

1. *ViewAxes* command, which allows the phase plane to be plotted, rather than plots of $x(i)$ vs t.
2. *Dir.field/flow* command, which allows trajectories to be plotted in the phase plane.
3. *Xi vs t:* which toggles between plots of $x(i)$ vs t for a multi-component system. It also allows auxilliary variables such as the potential and kinetic energy, which are defined in the *.ode* file, to be plotted.
4. *Initialconds:* with *mouse* command, which allows one to click onto a point in the phase plane and then integrate forward in time from that point. Freezing each one of the resulting trajectories allows a hardcopy to be obtained.
5. *Sing pts:* Allows one to compute equilibrium points, and to follow the invariant manifolds emanating from the equilibrium point in the phase plane.
6. **Nullclines:** will plot the nullclines for you.