MATH 345: LAB 4: TWO MECHANICS PROBLEMS

Purpose: The purpose of this lab is to use **xppaut** to explore the dynamics associated with the simple pendulum subject to damping. We will also study the motion of a glider as described by problem 6.5.14 in Strogatz's text.

The following ODE models a simple pendulum of length L subject to a linear damping force and gravity:

$$\theta^{''} + \mu\theta^{'} + \sin\theta = 0$$

Here $\mu \geq 0$ is a constant and we have non-dimensionalized the time variable so that we can set $g/L = 1$, without loss of generality.

Loading the ODE system: Please copy and paste the file *pend.ode* into your directory from my website. It has the form

```
# damped simple pendulum pend.ode
dthet/dt = thetp
dthetp/dt = -mu*thetp-sin(thet)
pe=(1-cos(thet))
ke=.5*(thetp**2)
aux P.E.=pe
aux K.E.=ke
aux T.E.=pe+ke
param mu=0.4
@ XP=T, YP=thet, XLO=0.0, XHI=20.0, YLO=-3.2, YHI=3.2
done
```

This defines the potential (P.E.), kinetic (K.E.) and the total (T.E.) energy as auxilliary variables, which can be plotted at the screen.

Calling the Program: To invoke the program follow the steps indicated

1. In your main directory, type *xppaut* and hit *return.* (this calls the executable)
2. You will then get a window that appears where you can click on the file harvest.ode that you wish to load.
3. At this stage XPPAUT opens a big window with MANY options should appear. We will examine only a few of the large number of options in this lab.

In this lab we will use the commands illustrated in Lab 1 and we will focus on some extra features of **xppaut** for multi-component including:

1. *ViewAxes* command, which allows the phase plane to be plotted, rather than plots of $x(i)$ vs t.

2. *Dir.field/flow* command, which allows trajectories to be plotted in the phase plane.
3. *Xi vs t:* which toggles between plots of $x(i)$ vs t for a multi-component system. It also allows auxilliary variables such as the potential and kinetic energy, which are defined in the *.ode* file, to be plotted.
4. *Initialconds:* with *mouse* command, which allows one to click onto a point in the phase plane and then integrate forward in time from that point. Freezing each one of the resulting trajectories allows a hardcopy to be obtained.
5. *Sing pts:* Allows one to compute equilibrium points, and to follow the invariant manifolds emanating from the equilibrium point in the phase plane.

1. TUTORIAL OF NEW COMMANDS

Computing Trajectories: The window should initially be *Theta* versus *t*. Click on the par icon and change the damping coefficient so that $\mu = 0$. Click on *Initialconds range* option and choose *start=0.0* and *end=3.1* and take *steps=10.* What does this say about the graph of the period of oscillation versus the amplitude? *Erase* the picture and Freeze a few of these trajectories one at a time using the *Initialconds new* command (take a trajectory with initial condition 3.1). Save the trajectories in a *.ps* file using the *Graphic stuff* command for plotting.

Now choose an initial condition of $\theta = 3.1$, $\theta^{'} = 0$ using the *Initialconds (new)* command. Then, click on *Xi vs t* and enter *THETP* at the command line. Notice that it then plots $\theta^{'}$ versus t corresponding to the trajectory θ versus t computed earlier. Click on *Xi vs t* once again and type *THET* to get back to the orginal plot. Notice that this option automatically fits the trajectory into the box (i.e. it changes the size of the window automatically to accomodate the plot). If you click on *Xi vs t* and type on the command line any one of the auxilliary variables *P.E.*, *K.E.*, or *T.E.* you can view the potential, kinetic, and total energy for the corresponding trajectory. The total energy should be a constant since there is no damping. If you like you can change the size of the window by using either the *Window/Zoom Hi:* option or the *Viewaxes (2-D)* option. This will allow a re-scaling in the vertical direction by re-labelling *ylo* and *yhi.*

The Phase-Plane: Make sure $\mu = 0$ so that there is no damping. Click on the *Viewaxes* command and choose the *2D* option. Change the X-axis to *THET* and the Y-axis to *THETP.* Choose for the bounds: *xmin=-7.0, xmax=7.0, ymin=-4.0, ymax=4.0* and label the X and Y axis appropriately. Click on *ok.* This should set up the phase plane with $\theta^{'}$ on the vertical axis and θ on the horizontal. Click on *Dir.field/flow* and click on the *flow* option. Enter for *grid* 5 instead of 10. This will divide the phase plane into 25 squares and compute trajectories starting from a point inside each of the squares up to a time $t = 20$. Notice that the trajectories above the horizontal axes move to the right as t increases, whereas those trajectories below the horizontal axis move to the left as t increases. At any time you can stop the computing by typing ./ at the terminal. The resulting phase plane will *NOT* be saved for plotting, but it does give a good indication of the global behavior.

To freeze trajectories to get a plot for later, click on *Initialconds* command with the *mouse* option. This allows you to select an initial condition in the box with the left mouse button. A trajectory will be drawn from that point for increasing time up to a time $t = 20$. Save the trajectory by clicking on *Graphic Stuff* and chosing the *Freeze*, *Freeze* commands. Keep repeating this until you have enough curves to get a good idea of what the phase plane looks likes. *remember the flow*

is from right to left below the horizontal line. Label the axes using the *Text etc* command.
Singular Points: To determine the equilibrium points use the *IC's* icon to specify an initial guess. Choose *thet=3.14*, *thetp=0.0*. Click on *sing points* option. The eigenvalues will be displayed at the console as well as the equilibrium point and its stability. The following symbols are:

r+ is the number of positive real eigenvalues

r- is the number of negative real eigenvalues

c+ is the number of complex eigenvalues with positive real parts

c- is the number of complex eigenvalues with negative real parts

You can follow the separatrices emanating from the equilibrium point by clicking *yes* on computing the invariant manifold option and using the ESC command at the terminal to trace out both branches. A ./ command ends the computation. What do you get when you try to compute the eigevalues for the equilibrium point at the origin?

To quit: Click on the *File* command and then click on the *quit* option in the submenu and choose *yes*.

Printout: To get a printout of the file *yourfile.ps* simply type *lpr yourfile.ps* at the unix command line. You can pop open a unix window at any time to plot any *.ps* file.

2. THE SPECIFIC LAB PROJECT

1. **The Simple Pendulum:**

$$\theta^{''} + \mu\theta^{'} + \sin\theta = 0$$

   i) **Let** $\mu = 0$**:** What is the qualitative behavior of the period as a function of the initial angle θ_0, where $\theta(0) = \theta_0$ and $\theta^{'}(0) = 0$? Justify your answer with a plot. As $\theta_0 \to \pi$, how does the period diverge to infinity? (You will need to argue analytically here). Turn in a plot of the phase plane and comment on the behavior of the solution for different initial conditions and the corresponding behavior in the physical plane.

   ii) **Let** $\mu = .4$**:** Turn in plots of θ versus t and $T.E.$ versus t for different initial conditions. Turn in a plot of the phase plane. What is the behavior near the origin? What are the eigenvalues corresponding to the linearization near $\theta = 0$ and $\theta = \pi$. Explain the different types of behaviors in the physical plane as we vary the initial conditions.

2. **The Glider Problem:** As derived in class (see problem 6.5.14), the equations describing the motion of a glider are

$$v^{'} = -\sin\theta - Dv^2$$
$$\theta' = v^{-1}\left(-\cos\theta + v^2\right)$$

The trigonometric terms represent the effect of gravity and the v^2 terms model the effect of lift and drag. The file is on my website in *glider.ode*. Please copy and paste it into your directory.

   i) **Let** $D = 0$**:** Plot the phase-plane, show that $v^3 - 3v\cos\theta$ is a conserved quantity and interpret the results physically. What does the flight path of the glider look like?

   ii) Investigate the case of a positive drag D. Plot the phase plane in this case and intepret the results physically. What does the flight path look like now?