MATH 316/257 SOLUTIONS TO MIDTERM 2

PROBLEM 1

$$u_t = u_{xx} - 4u \quad \text{FOR} \quad 0 \le x \le \pi, \; t > 0$$

$$u_x(0,t) = 0, \quad u_x(\pi, t) = 1 \quad \text{FOR} \quad t > 0$$

$$u(x,0) = 1 + \cos(2x).$$

(i) FIND STEADY-STATE $u_s(x)$. IT SATISFIES

$$u_s'' - 4u_s = 0 \quad \text{IN} \quad 0 \le x \le \pi$$

$$u_s'(0) = 0, \quad u_s'(\pi) = 1.$$

WE GET $u_s(x) = A\cosh(2x) + B\sinh(2x)$

NOW $u_s'(0) = 0 \longrightarrow B = 0$

THEN $u_s'(\pi) = 1 \longrightarrow 2A\sinh(2\pi) = 1 \longrightarrow A = \dfrac{1}{2\sinh(2\pi)}$

THIS GIVES $u_s(x) = \dfrac{\cosh(2x)}{2\sinh(2\pi)}$.

(ii) WE EXPAND IN A FOURIER COSINE SERIES:

$$u_s(x) = \frac{\cosh(2x)}{2\sinh(2\pi)} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

THEN
$$a_n = \frac{2}{\pi}\int_0^{\pi} u_s(x)\cos(nx)\,dx \qquad n = 0, 1, 2, \ldots$$

THUS
$$a_n = \frac{1}{\pi}\,\frac{1}{\sinh(2\pi)}\int_0^{\pi} \cosh(2x)\cos(nx)\,dx$$

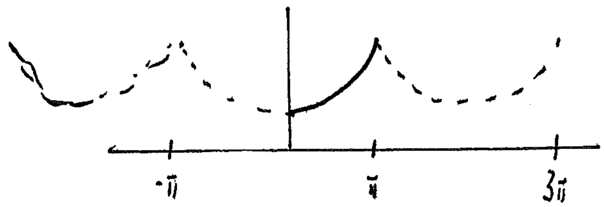
USING THE INTEGRAL

$$a_n = \frac{1}{\pi\sinh(2\pi)}\,\frac{2(-1)^n\sinh(2\pi)}{4+n^2} = \frac{2(-1)^n}{\pi(4+n^2)}$$

WE OBTAIN $a_0 = 1/2\pi$, $\quad a_n = 2(-1)^n/\pi[4+n^2]$ FOR $n = 1, 2, \ldots$

WE OBTAIN
$$\frac{\cosh(2x)}{2\sinh(2\pi)} = \frac{1}{4\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4+n^2}\cos(nx) \qquad (1)$$

(iii) NOW IN (1) EXTEND u_s TO BE EVEN ON $-\pi < x < 0$ AND THEN EXTEND PERIODICALLY. WE GET



THUS BY THE FOURIER CONVERGENCE THEOREM (1) CONVERGES TO $u_s(x)$ AT EACH x. SET $x = \pi$. THEN

$$\frac{\cosh(2\pi)}{2\sinh(2\pi)} = \frac{1}{4\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi)}{4+n^2}$$

SINCE $\cos(n\pi) = (-1)^n$ WE GET WITH $\cos(n\pi)(-1)^n = (-1)^{2n} = 1, \ \forall n$ THAT

$$\frac{\pi}{2}\left[\frac{\coth(2\pi)}{2} - \frac{1}{4\pi}\right] = \sum_{n=1}^{\infty} \frac{1}{4+n^2}$$

SO $$\sum_{n=1}^{\infty} \frac{1}{4+n^2} = \frac{\pi}{4}\coth(2\pi) - \frac{1}{8}$$

<u>REMARK</u> <u>(NOT FOR TEST)</u> IF YOU PUT $x = \pi/2$ IN (1) YOU GET

$$\frac{\cosh(\pi)}{2\sinh(2\pi)} = \frac{1}{4\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{4+n^2}\cos\left(\frac{n\pi}{2}\right)$$

BUT $\cos\left(\frac{n\pi}{2}\right) = 0$ IF $n = 1,3,5$

$$= \frac{1}{4\pi} + \frac{2}{\pi}\sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \frac{(-1)^n}{4+n^2}\cos\left(\frac{n\pi}{2}\right) = \frac{1}{4\pi} + \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^{2m}}{4+4m^2}\cos(m\pi)$$

(WITH $n = 2m$)

$$\frac{\cosh(\pi)}{2\left[2\sinh(\pi)\cosh(\pi)\right]} = \frac{1}{4\pi} + \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^m}{4+4m^2}$$

SO $$\frac{1}{4\sinh\pi} = \frac{1}{4\pi} + \frac{1}{2\pi}\sum_{m=1}^{\infty} \frac{(-1)^m}{m^2+1}.$$

WE GET $$\sum_{m=1}^{\infty} \frac{(-1)^m}{m^2+1} = \frac{\pi}{2\sinh(\pi)} - \frac{1}{2}.$$

(iii) WE PUT $u = u_s(x) + v(x,t)$ INTO THE PDE:

$$(u_s + v)_t = (u_s + v)_{xx} - 4(u_s + v) = \underbrace{u_s'' - 4u_s}_{\to 0 \leftarrow} + v_{xx} - 4v$$

SO
$$v_t = v_{xx} - 4v \quad \text{ON} \quad 0 < x < \pi, \ t > 0$$
$$v_x(0,t) = 0, \quad v_x(\pi, t) = 0$$
$$v(x,0) = 1 + \cos(2x) - u_s(x).$$

NOW SEPARATE VARIABLES $v = T(t)\Phi(x)$. THEN

$$T'\Phi = T\Phi'' - 4T\Phi \quad \text{SO} \quad \frac{T' + 4T}{T} = \frac{\Phi''}{\Phi} = -\lambda. \qquad T' = -(\lambda + 4)T.$$

WE HAVE
$$\Phi'' + \lambda\Phi = 0 \qquad \text{SO} \qquad \lambda_0 = 0, \ \Phi_0 = 1$$
$$\Phi'(0) = \Phi'(\pi) = 0 \qquad\qquad \lambda_n = n^2, \ \Phi_n(x) = \cos(nx)$$

IF $\lambda_0 = 0$, $T_0' = -4T_0 \rightarrow T_0 = e^{-4t}$

IF $\lambda_n = n^2$, $T_n' = -(4+n^2)T_n \rightarrow T_n = e^{-(4+n^2)t}$

THUS
$$v(x,t) = c_0 e^{-4t} + \sum_{n=1}^{\infty} c_n e^{-(4+n^2)t} \cos(nx).$$

NOW
$$v(x,0) = c_0 + \sum_{n=1}^{\infty} c_n \cos(nx) = 1 + \cos(2x) - u_s(x).$$

SO
$$c_0 + \sum_{n=1}^{\infty} c_n \cos(nx) = 1 + \cos(2x) - \frac{1}{4\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{4+n^2}\cos(nx)$$

WE CONCLUDE THAT

$$c_0 = 1 - 1/4\pi, \qquad c_n = -\frac{2}{\pi}\frac{(-1)^n}{4+n^2}, \quad n = 1,3,5,6,7,\ldots$$

AND
$$c_2 = 1 - \frac{2}{\pi}\frac{(-1)^n}{4+n^2} = 1 - \frac{1}{4\pi} \quad \text{IF} \quad n = 2.$$

THEN
$$u(x,t) = u_s(x) + \left(1 - \frac{1}{4\pi}\right)e^{-4t} + \sum_{n=1}^{\infty} c_n e^{-(4+n^2)t}\cos(nx),$$

WHERE $u_s(x) = \cosh(2x) / 2\sinh(2x)$.

v) LET $M(t) = \int_0^{\pi} u(x,t)\,dx$.

INTEGRATE THE PDE $\int_0^{\pi} u_t\,dx = \int_0^{\pi} u_{xx}\,dx - 4\int_0^{\pi} u$

SO $$\frac{d}{dt}\int_0^{\pi} u\,dx = u_x\Big|_0^{\pi} - 4\int_0^{\pi} u\,dx.$$

BUT $u_x(\pi,t) = 1$ AND $u_x(0,t) = 0$ SO

$$\frac{dM}{dt} + 4M = 1.$$

NOW THE SOLUTION IS

$$M = \frac{1}{4} + \alpha e^{-4t}.$$

BUT $M(0) = \int_0^{\pi} u(x,0)\,dx = \int_0^{\pi}\left(1 + \cos(2x)\right)dx = \pi.$

SO $\pi = \frac{1}{4} + \alpha \rightarrow \alpha = \pi - 1/4.$

SO $$M(t) = \frac{1}{4} + \left(\pi - 1/4\right)e^{-4t}$$

WITH $M(t) \rightarrow 1/4$ AS $t \rightarrow \infty$.

PROBLEM 2

$$u_{tt} = 4u_{xx} + xt \qquad 0 < x < \pi, \quad t > 0$$

$$u(0,t) = 0, \quad u(\pi,t) = 1$$

$$u(x,0) = x/\pi, \quad u_t(x,0) = 1$$

LET $w(x)$ SATISFY $\left.\begin{array}{l} w_{xx} = 0 \\ w(0) = 0, \quad w(\pi) = 1 \end{array}\right\} \to w = x/\pi$

NOW PUT $u = x/\pi + v(x,t)$ TO OBTAIN

$$(1) \quad \left\{ \begin{array}{l} v_{tt} = 4v_{xx} + xt \\ v(0,t) = 0, \quad v(\pi,t) = 0 \\ v(x,0) = u(x,0) - x/\pi = 0, \quad v_t(x,0) = 1. \end{array} \right.$$

NOW EXPAND IN TERMS OF EIGENFUNCTIONS OF HOMOGENEOUS PROBLEM

$\Phi_n(x) = \sin(nx)$.

THUS $$v(x,t) = \sum_{n=1}^{\infty} c_n(t) \sin(nx).$$

WE SUBSTITUTE $$\sum_{n=1}^{\infty} c_n'' \sin(nx) = \sum_{n=1}^{\infty} 4 c_n \frac{d^2}{dx^2} \sin(nx) + xt$$

SO $$\sum_{n=1}^{\infty} (c_n'' + 4n^2 c_n) \sin(nx) = xt$$

NOW EXPAND $$xt = \sum_{n=1}^{\infty} f_n(t) \sin(nx) \qquad f_n(t) = \frac{2}{\pi}\int_0^{\pi} xt \sin(nx)\, dx$$

THUS $$f_n(t) = t\,\gamma_n \quad \text{WHERE} \quad \gamma_n \equiv \frac{2}{\pi}\int_0^{\pi} x \sin(nx)\, dx. \quad (2)$$

WE GET $$\sum_{n=1}^{\infty} (c_n'' + 4n^2 c_n) \sin(nx) = \sum_{n=1}^{\infty} t\,\gamma_n \sin(nx).$$

WE CONCLUDE THAT

$$C_n'' + 4n^2 C_n = \gamma_n t \qquad \text{FOR} \quad n \geq 1$$

TO GET INITIAL CONDITIONS FOR THIS WE USE

$$V(x,0) = \sum_{n=1}^{\infty} C_n(0) \sin(nx) = 0 \quad \rightarrow \quad C_n(0) = 0.$$

$$V_t(x,0) = \sum_{n=1}^{\infty} C_n'(0) \sin(nx) = 1 \quad \rightarrow \quad C_n'(0) = B_n \equiv \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx. \qquad (3)$$

SO WE MUST SOLVE

$$\begin{cases} C_n'' + 4n^2 C_n = \gamma_n t \qquad \text{FOR} \quad n \geq 1 \\ C_n(0) = 0, \quad C_n'(0) = B_n \end{cases}$$

WITH γ_n, B_n GIVEN IN (2) AND (3)

THE SOLUTION IS $C_n = A\cos(2nt) + B\sin(2nt) + \dfrac{\gamma_n}{4n^2} t.$

NOW $C_n(0) = 0 \rightarrow A = 0$

$$C_n'(0) = B_n \quad \rightarrow \quad 2nB + \frac{\gamma_n}{4n^2} = B_n \quad \text{SO} \quad B = \frac{1}{2n}\left[B_n - \frac{\gamma_n}{4n^2}\right].$$

WE CONCLUDE THAT $C_n(t) = \dfrac{1}{2n}\left[B_n - \dfrac{\gamma_n}{4n^2}\right]\sin(2nt) + \dfrac{\gamma_n}{4n^2} t$

AND OUR SOLUTION IS $u(x,t) = x/\pi + \displaystyle\sum_{n=1}^{\infty} C_n(t)\sin(nx)$

FINALLY, WE CALCULATE B_n AND γ_n.

$$B_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx = -\frac{2}{\pi n}\cos(nx)\Big|_0^{\pi} = -\frac{2}{\pi n}\left[(-1)^n - 1\right] = \frac{2}{\pi n}\left[1-(-1)^n\right].$$

$$\gamma_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx = \frac{2}{\pi}\Big[-\frac{x}{n}\cos(nx)\Big|_0^{\pi} + \frac{1}{n}\underbrace{\int_0^{\pi}\cos(nx)\,dx}_{=0}\Big] = -\frac{2\pi(-1)^n}{\pi n} = -\frac{2}{n}(-1)^n$$

$u = x,\ du = dx$

$dv = \sin(nx)\,dx \quad v = -\frac{1}{n}\cos(nx)$

SO $\gamma_n = -\dfrac{2}{n}(-1)^n.$