

Lecture 30: Sturm-Liouville Problems involving the Cauchy-Euler Equation - Applications

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In this lecture we look at eigenvalue problems involving equidimensional or Cauchy-Euler differential operators.

Key Concepts: Eigenvalue Problems, Sturm-Liouville Boundary Value Problems; Cauchy-Euler Equations; Equidimensional equations.

Reference Section: Boyce and Di Prima Section 11.1 and 11.2

30 Variable coefficient BVP - eigenfunctions involving solutions to the Euler Equation:

Example 30.1 *Eigenfunctions involving solutions to an Euler Equation:*

$$\begin{aligned}(x^2\phi')' + \lambda\phi &= 0 & 1 < x < 2 \\ \phi(1) = 0, \phi(2) &= 0 \\ x^2\phi'' + 2x\phi' + \lambda\phi &= 0 & \text{A Cauchy-Euler Eq.}\end{aligned}\tag{30.1}$$

Let

$$\phi(x) = x^r \quad r(r-1) + 2r + \lambda = r^2 + r + \lambda = 0.\tag{30.2}$$

Therefore

$$r = \frac{-1 \pm \sqrt{1 - 4\lambda}}{2} = r_1, r_2.\tag{30.3}$$

Case I: $\lambda = \frac{1}{4}$:

$$\phi(x) = c_1x^{-\frac{1}{2}} + c_2x^{-\frac{1}{2}} \log x\tag{30.4}$$

$$\phi(1) = c_1 = 0 \quad \phi(2) = c_22^{-\frac{1}{2}} \log 2 = 0 \Rightarrow c_2 = 0\tag{30.5}$$

so there is no nontrivial eigenfunction for $\lambda = 1/4$.

Case II: $\lambda \neq \frac{1}{4}$:

$$\phi(x) = c_1x^{r_1} + c_2x^{r_2}\tag{30.6}$$

$$\phi(1) = c_1 + c_2 = 0 \quad c_2 = -c_1\tag{30.7}$$

$$\phi(2) = c_1(2^{r_1} - 2^{r_2}) = 0\tag{30.8}$$

$$\begin{aligned}2^{r_1 - r_2} &= 1 \\ e^{(r_1 - r_2) \ln 2} &= 1 = e^{2\pi in} \\ (r_1 - r_2) \ln 2 &= 2\pi in \\ r_1 - r_2 &= \sqrt{1 - 4\lambda} = 2\pi ni / \ln(2)\end{aligned}\tag{30.9}$$

Thus to obtain nontrivial solutions we require $1 - 4\lambda < 0$ which implies $\lambda > \frac{1}{4}$. Thus for $\lambda > \frac{1}{4}$

$$\sqrt{1 - 4\lambda} = i\sqrt{4\lambda - 1} = 2\pi ni / \ln(2). \quad (30.10)$$

The Eigenvalues are:

$$\lambda_n = \frac{1}{4} + \frac{\pi^2 n^2}{[\ln(2)]^2}, \quad 4\lambda_n - 1 = \frac{4\pi^2 n^2}{[\ln(2)]^2} = (2\beta_n)^2 \quad \beta_n = (n\pi / \ln 2). \quad (30.11)$$

The corresponding roots r_1 and r_2 are as follows

$$(r_1)_n = -\frac{1}{2} + i\beta_n \quad \text{and} \quad (r_2)_n = -\frac{1}{2} - i\beta_n \quad (30.12)$$

$$\phi_n(x) = c_n x^{-\frac{1}{2}} (x^{i\beta_n} - x^{-i\beta_n}) \quad (30.13)$$

$$= c_n x^{-\frac{1}{2}} [e^{i\beta_n \ln x} - e^{-i\beta_n \ln x}] \quad (30.14)$$

$$= d_n x^{-\frac{1}{2}} \sin(\beta_n \ln x) \quad (30.15)$$

$$= d_n x^{-\frac{1}{2}} \sin \left[n\pi \frac{\ln x}{\ln 2} \right] \quad (30.16)$$

Now let us consider expanding a function $f(x)$ in terms of a 'Fourier Series' of these new eigenfunctions in the following form

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) = \sum_{n=1}^{\infty} c_n x^{-1/2} \sin \left(\frac{n\pi \ln x}{\ln 2} \right) \quad (30.17)$$

In order to determine the coefficients c_n we project the function $f(x)$ onto the basis functions $\phi_n(x)$ as follows:

$$\int_1^2 f(x) x^{-1/2} \sin \left(\frac{m\pi \ln x}{\ln 2} \right) dx = \sum_{n=1}^{\infty} c_n \int_1^2 x^{-1/2} \sin \left(\frac{m\pi \ln x}{\ln 2} \right) x^{-1/2} \sin \left(\frac{n\pi \ln x}{\ln 2} \right) dx$$

The integrals under the summation can be evaluated by making the simple substitution $z = \ln x$ so that $dz = \frac{dx}{x}$, in which case:

$$\int_1^2 \sin \left(\frac{m\pi \ln x}{\ln 2} \right) \sin \left(\frac{n\pi \ln x}{\ln 2} \right) \frac{dx}{x} = \int_0^{\ln 2} \sin \left(\frac{m\pi z}{\ln 2} \right) \sin \left(\frac{n\pi z}{\ln 2} \right) dz = \delta_{mn} \frac{\ln 2}{2}$$

Substituting this result into (30.17) we obtain the following expression for the Fourier coefficients c_n :

$$c_n = \frac{2}{\ln 2} \int_1^2 f(x) x^{-1/2} \sin \left(\frac{n\pi \ln x}{\ln 2} \right) dx$$

Example 30.2 A variable coefficient Heat Conduction Problem with Cauchy-Euler Eigenfunctions:

$$\begin{aligned} u_t &= D(x^2 u_x)_x - u & 1 < x < 2 \quad t > 0 \\ u(1, t) &= 0 = u(2, t) & u(x, 0) &= f(x). \end{aligned} \tag{30.18}$$

Let

$$u(x, t) = X(x)T(t) \tag{30.19}$$

$$\frac{\dot{T}(t)}{DT(t)} = \frac{(x^2 X')'}{X} - 1 \tag{30.20}$$

$$\frac{\dot{T}(t)}{DT(t)} + 1 = \frac{(x^2 X')'}{X} = -\lambda \tag{30.21}$$

$$\left. \begin{aligned} \dot{T} + D(1 + \lambda)T &= 0 \\ (x^2 X')' + \lambda X &= 0 \\ X(1) = 0 = X(2) \end{aligned} \right\} \begin{aligned} T(t) &= ce^{-D(1+\lambda)t} \\ \lambda_n &= \frac{1}{4} + \frac{(\pi n)^2}{[\ln(2)]^2}; X_n(x) = x^{-\frac{1}{2}} \sin\left(n\pi \frac{\ln x}{\ln 2}\right) \\ n &= 1, 2, \dots \end{aligned} \tag{30.22}$$

$$u(x, t) = x^{-\frac{1}{2}} \sum_{n=1}^{\infty} c_n e^{-D(1+\lambda_n)t} \sin\left(n\pi \frac{\ln x}{\ln 2}\right) \tag{30.23}$$

$$f(x) = u(x, 0) = x^{-\frac{1}{2}} \sum_{n=1}^{\infty} c_n \sin\left(n\pi \frac{\ln x}{\ln 2}\right) \tag{30.24}$$

where

$$c_n = \frac{2}{\ln 2} \int_1^2 f(x) x^{-1/2} \sin\left(\frac{n\pi \ln x}{\ln 2}\right) dx$$

31 Solving Laplace's equation using Cauchy-Euler eigenfunctions

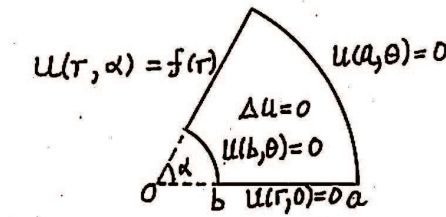


FIGURE 1. Annular sector subtending an arc of α radians between the radii a and b : $1 = b < a = 2$

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad 1 < r < 2, \quad 0 < \theta < \alpha \tag{31.1}$$

$$u(r, 0) = 0, \quad u(r, \alpha) = f(r) \tag{31.2}$$

$$u(1, \theta) = 0, \quad u(2, \theta) = 0 \tag{31.3}$$

$$u(r, \theta) = R(r)\Theta(\theta) \tag{31.4}$$

$$\frac{r^2 R'' + r R'}{R(r)} = -\frac{\Theta''}{\Theta} = -\lambda^2 \quad (\text{because of Homog. BC}) \tag{31.5}$$

$$\Theta: \quad \Theta'' - \lambda^2 \Theta = 0 \quad \Theta = C \cosh \lambda \theta + D \sinh \lambda \theta$$

$$\Theta(0) = 0 \quad \Theta(0) = C = 0 \Rightarrow \Theta(\theta) = D \sinh \lambda \theta$$

R: $r^2 R'' + rR' + \lambda^2 R = 0(\star)$, $R(1) = 0 = R(2)$ Although we can easily see that dividing through by r we can reduce (\star) to S-L form, let us use the integrating factor

$$\mu(r) = \frac{1}{P} e^{\int \frac{Q}{P} dr} = \frac{1}{r^2} e^{\int \frac{r}{r^2} dr} = \frac{1}{r^2} e^{\ln r} = \frac{1}{r}. \quad (31.6)$$

Therefore $-\frac{1}{r} \cdot (\star)$ and some rearrangement implies:

$$-rR'' - R' = -(rR')' = \frac{\lambda^2}{r} R$$

Now let us look for Eigenvalues and Eigenfunctions to (\star) . Let

$$R(r) = r^\gamma \Rightarrow \gamma(\gamma - 1) + \gamma + \lambda^2 = \gamma^2 + \lambda^2 = 0 \quad \gamma = \pm i\lambda. \quad (31.7)$$

$$R(r) = c_1 r^{i\lambda} + c_2 r^{-i\lambda} \quad r^{i\lambda} = e^{i\lambda \ln r} \quad (31.8)$$

$$= A \cos(\lambda \ln r) + B \sin(\lambda \ln r) \quad (31.9)$$

$$R(1) = A \cos[\lambda(\ln 1)] + B \sin(\lambda \ln 1) = A = 0 \quad (31.10)$$

$$R(2) = B \sin[\lambda \ln 2] = 0 \Rightarrow \lambda_n \ln 2 = n\pi \quad n = 1, 2, \dots \quad (31.11)$$

and the corresponding Eigenfunctions are $R_n = \sin\left(n\pi \frac{\ln r}{\ln 2}\right)$. Therefore

$$u(r, \theta) = \sum_{n=1}^{\infty} B_n \sinh(\lambda_n \theta) \sin(\lambda_n \ln r). \quad (31.12)$$

Now match BC:

$$f(r) = u(r, \alpha) = \sum_{n=1}^{\infty} B_n \sinh(\lambda_n \alpha) \sin\left(n\pi \frac{\ln r}{\ln 2}\right) \quad (31.13)$$

Now making the substitution $x = \ln r$ so that $dx = \frac{dr}{r}$ we can reduce the orthogonality integrals for the Fourier coefficients to the form:

$$\int_1^2 \frac{1}{r} \sin\left(\frac{m\pi \ln r}{\ln 2}\right) \sin\left(\frac{n\pi \ln r}{\ln 2}\right) dr = \int_0^{\ln 2} \sin\left(\frac{m\pi x}{\ln 2}\right) \sin\left(\frac{n\pi x}{\ln 2}\right) dx = \begin{cases} 0 & m \neq n \\ \frac{\ln 2}{2} & m = n \end{cases}. \quad (31.14)$$

Therefore

$$B_n = \frac{2}{\ln 2 \sinh\left(\frac{n\pi \alpha}{\ln 2}\right)} \int_1^2 \frac{f(r)}{r} \sin\left(\frac{n\pi \ln r}{\ln 2}\right) dr. \quad (31.15)$$