Lecture 28: Sturm-Liouville Boundary Value Problems

(*Compiled 3 March 2014*)

In this lecture we abstract the eigenvalue problems that we have found so useful thus far for solving the PDEs to a general class of boundary value problems that share a common set of properties. The so-called *Sturm-Liouville Problems* define a class of eigenvalue problems, which include many of the previous problems as special cases. The $S - L$ Problem helps to identify those assumptions that are needed to define an eigenvalue problems with the properties that we require.

Key Concepts: Eigenvalue Problems, Sturm-Liouville Boundary Value Problems; Robin Boundary conditions.

Reference Section: Boyce and Di Prima Section 11.1 and 11.2

30 Boundary value problems and Sturm-Liouville theory:

30.1 Eigenvalue problem summary

- We have seen how useful eigenfunctions are in the solution of various PDEs.
- The eigenvalue problems we have encountered thus far have been relatively simple

I: The Dirichlet Problem:

$$\left.\begin{array}{c} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X(L) \end{array}\right\} \Longrightarrow \left\{ \begin{array}{c} \lambda_n = \frac{n\pi}{L},\ n = 1, 2, \ldots \\ X_n(x) = \sin\left(\frac{n\pi x}{L}\right) \end{array}\right.$$

II: The Neumann Problem:

$$\left.\begin{array}{c} X'' + \lambda^2 X = 0 \\ X'(0) = 0 = X'(L) \end{array}\right\} \Longrightarrow \left\{ \begin{array}{c} \lambda_n = \frac{n\pi}{L},\ n = 0, 1, 2, \ldots \\ X_n(x) = \cos\left(\frac{n\pi x}{L}\right) \end{array}\right.$$

III: The Periodic Boundary Value Problem:

$$\left.\begin{array}{c} X'' + \lambda^2 X = 0 \\ X(-L) = 0 = X(L) \\ X'(-L) = 0 = X'(L) \end{array}\right\} \Longrightarrow \left\{ \begin{array}{c} \lambda_n = \frac{n\pi}{L},\ n = 0, 1, 2, \ldots \\ X_n(x) \in \left\{1, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\} \end{array}\right.$$

IV: Mixed Boundary Value Problem A:

$$\left.\begin{array}{c} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X'(L) \end{array}\right\} \Longrightarrow \left\{ \begin{array}{c} \lambda_k = \frac{(2k+1)\pi}{2L},\ k = 0, 1, 2, \ldots \\ X_n(x) = \sin\left(\frac{(2k+1)\pi}{2L} x\right) \end{array}\right.$$

V: Mixed Boundary Value Problem B:

$$\left.\begin{array}{c} X'' + \lambda^2 X = 0 \\ X'(0) = 0 = X(L) \end{array}\right\} \Longrightarrow \left\{ \begin{array}{c} \lambda_k = \frac{(2k+1)\pi}{2L},\ k = 0, 1, 2, \ldots \\ X_n(x) = \cos\left(\frac{(2k+1)\pi}{2L} x\right) \end{array}\right.$$

30.2 The regular Sturm-Liouville problem:

Consider the the following two-point boundary value problem

$$\begin{array}{l} \big(p(x)y'\big)' - q(x)y + \lambda r(x)y = 0 \quad 0 < x < \ell \\ \alpha_1 y(0) + \alpha_2 y'(0) = 0 \quad \beta_1 y(\ell) + \beta_2 y'(\ell) = 0 \end{array} \tag{30.1}$$

where p, p', q and r are continuous on $0 \le x \le \ell$ and $p(x) \ge 0$ and $r(x) > 0$ on $0 \le x \le \ell$.

We define the Sturm-Liouville eigenvalue problem as:

$$\left.\begin{array}{l} \mathcal{L}y = \lambda r y \quad \text{where} \quad \mathcal{L}y = -(py')' + qy \\ \alpha_1 y(0) + \alpha_2 y'(0) = 0 \quad \text{and} \quad \beta_1 y(\ell) + \beta_2 y'(\ell) = 0 \\ p(x) > 0 \text{ and } r(x) > 0. \end{array}\right\} \quad \text{SL} \tag{30.2}$$

Remark 1 Note:

(1) If $p = 1$, $q = 0$, $r = 1$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = 1$, $\beta_2 = 0$ we obtain Problem (I) above whereas if $p = 1$, $q = 0$, $r = 1$, $\alpha_1 = 0$, $\alpha_2 = 1$, $\beta_1 = 0$, $\beta_2 = 1$, we obtain Problem (II) above. Notice that the boundary conditions for these two problems are specified at separate points and are called *separated BC*. The periodic BC $X(0) = X(2\pi)$ are not separated so that Problem (III) is not technically a SL Problem.

(2) If $p > 0$ and $r > 0$ and $\ell < \infty$ then the SL Problem is said to be regular. If $p(x)$ or $r(x)$ is zero for some x or the domain is $[0, \infty)$ then the problem is singular.

(3) There is no loss of generality in the form of $\mathcal{L}y = -(py') + qy$ since it is possible to convert a general 2nd order eigenvalue problem

$$-P(x)y'' - Q(x)y' + R(x)y = \lambda y \tag{30.3}$$

to this form by multiplying by an integrating factor $\mu(x)$

$$-\mu(x)P(x)y'' - \mu Q(x)y' + \mu(x)R(x)y = \lambda\mu(x)y \tag{30.4}$$

but expanding the differential operator we obtain

$$\mathcal{L}y = -py'' - p'y' + qy = \lambda r y. \tag{30.5}$$

Thus comparing (30.5) and (30.4) we can make the following identifications: $p = \mu P$ and $p' = \mu Q \Rightarrow p' = \mu' P + \mu P' = \mu Q$ which is a linear 1st order ODE for μ with integrating factor $exp(\int \frac{P'}{P} - \frac{Q}{P}\, dx)$

$$\mu' + \left(\frac{P'}{P} - \frac{Q}{P}\right)\mu = 0 \Rightarrow \left[P e^{-\int \frac{Q}{P}\, dx}\mu\right]' = 0 \quad \boxed{\mu = \frac{e^{\int \frac{Q}{P}\, dx}}{P}}. \tag{30.6}$$

Example 30.1 *Reducing a boundary value problem to SL form:*

$$\phi'' + x\phi' + \lambda\phi = 0 \tag{30.7}$$

$$\phi(0) = 0 = \phi \tag{30.8}$$

We bring (30.7) into SL form by multiplying by the integrating factor

$$
\begin{aligned}
&\mu = \frac{1}{P}e^{\int \frac{Q}{P}\,dx} = e^{\int x\,dx} = e^{x^2/2}, \quad P(x) = 1, \quad Q(x) = x, \quad R(x) = 1. \\
&e^{x^2/2}\phi'' + e^{x^2/2}x\phi' + \lambda e^{x^2/2}\phi = 0 \\
&\qquad -\left(e^{x^2/2}\phi'\right)' = \lambda e^{x^2/2}\phi \\
&p(x) = e^{x^2/2} \quad r(x) = e^{x^2/2}
\end{aligned}
\tag{30.9}
$$

Example 30.2 *Convert the equation* $-y'' + x^4y^1 = \lambda y$ *to SL form*

$$P = 1, \quad Q = -x^4, \quad \mu = e^{-\int x^4\,dx} = e^{-x^5/5} \tag{30.10}$$

$$\textit{Therefore } - e^{-x^5/5}y'' + e^{-x^5/5}x^4y' = \lambda e^{-x^5/5} \tag{30.11}$$

$$-\left(e^{-x^5/5}y'\right)' = \lambda e^{-x^5/5}y. \tag{30.12}$$

30.3 Properties of SL Problems

(1) **Eigenvalues**:

- **(a)** The eigenvalues λ are all real.
- **(b)** There are an ∞ # of eigenvalues λ_j with $\lambda_j < \lambda_2 < \ldots < \lambda_j \to \infty$ as $j \to \infty$.
- **(c)** $\lambda_j > 0$ provided $\dfrac{\alpha_1}{\alpha_2} < 0$, $\dfrac{\beta_1}{\beta_2} > 0$ $q(x) > 0$.

(2) **Eigenfunctions**: For each λ_j there is an eigenfunction $\phi_j(x)$ that is unique up to a multiplicative const. and which satisfy:

- **(a)** $\phi_j(x)$ are real and can be normalized so that $\int\limits_0^\ell r(x)\phi_j^2(x)\,dx = 1$.
- **(b)** The eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the weight function $r(x)$:

$$\int\limits_0^\ell r(x)\phi_j(x)\phi_K(x)\,dx = 0 \quad j \neq k. \tag{30.13}$$

- **(c)** $\phi_j(x)$ has exactly $j-1$ zeros on $(0, \ell)$.

(3) **Expansion Property**: $\{\phi_j(x)\}$ are complete if $f(x)$ is piecewise smooth then

$$
\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} c_n\phi_n(x) \\
\text{where} \qquad c_n &= \frac{\int_0^\ell r(x)f(x)\phi_n(x)\,dx}{\int_0^\ell r(x)\phi_n^2(x)\,dx}
\end{aligned}
\tag{30.14}
$$

Example 30.3 Robin Boundary Conditions:

$$X'' + \lambda X = 0, \qquad \lambda = \mu^2$$
$$X'(0) = h_1 X(0), \quad X'(\ell) = -h_2 X(\ell) \tag{30.15}$$

where $h_1 \geq 0$ *and* $h_2 \geq 0$.

$$X(x) = A\cos\mu x + B\sin\mu x \tag{30.16}$$

$$X'(x) = -A\mu\sin\mu x + B\mu\cos\mu x \tag{30.17}$$

BC 1: $X'(0) = B\mu = h_1 X(0) = A \quad A = B\mu/h_1$.

BC 2: $X'(\ell) = -A\mu\sin(\mu\ell) + B\mu\cos(\mu\ell) = -h_2 X(\ell) = -h_2[A\cos\mu\ell + B\sin\mu\ell]$

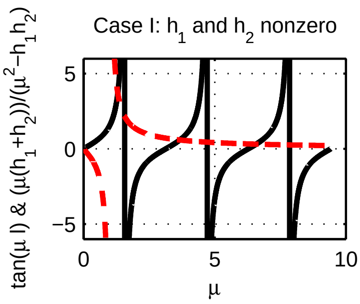
$$\Rightarrow B\left[-\frac{\mu^2}{h_1}\sin(\mu\ell) + \mu\cos(\mu\ell)\right] = -Bh_2\left[\frac{\mu}{h_1}\cos\mu\ell + \sin\mu\ell\right] \tag{30.18}$$

$$B\left\{\left(-\frac{\mu^2}{h_1} + h_2\right)\sin\mu\ell + \left(\mu + \frac{h_2}{h_1}\mu\right)\cos\mu\ell\right\} = 0. \tag{30.19}$$

Therefore

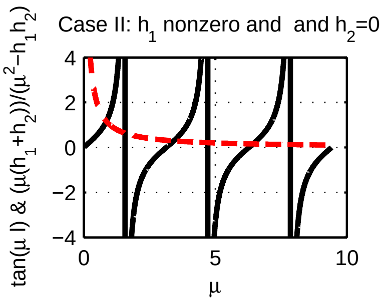
$$\tan(\mu\ell) = \left[\frac{\mu(h_1 + h_2)}{\mu^2 - h_1 h_2}\right]. \tag{30.20}$$

Case I: $\mathbf{h_1}$ and $\mathbf{h_2 \neq 0}$



$X_n = \frac{\mu_n}{h_1}\cos\mu_n x + \sin\mu_n x$, and $\mu_n \sim n\pi/\ell$ as $n \to \infty$

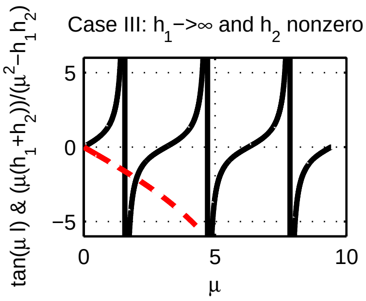
Case II: $\mathbf{h_1 \neq 0}$ and $\mathbf{h_2 = 0}$



$$X_n = \frac{\mu_n}{h_1}\cos\mu_n x + \sin\mu_n x \tag{30.21}$$

$$= \frac{\cos\mu_n(\ell - x)}{\sin\mu_n\ell} \tag{30.22}$$

Case III: $\mathbf{h_1 \to \infty \quad h_2 \neq 0}$



$$X_n = \sin(\mu_n x) \tag{30.23}$$

$$\mu_n \sim \left[\left(\frac{2n+1}{2}\right)\frac{\pi}{\ell}\right] \qquad n = 0, 1, 2, \ldots \text{ as } n \to \infty \tag{30.24}$$

30.4 Appendix: Some proofs for Sturm-Liouville Theory

30.4.1 *Lagrange's Identity:*

$$\int_0^\ell (v\mathcal{L}u - u\mathcal{L}v)\,dx = -p(x)u'v|_0^\ell + p(x)uv'|_0^\ell.$$

Proof: Let u and v be any sufficiently differentiable functions, then

$$\int_0^\ell v\mathcal{L}u\,dx = \int_0^\ell v\left\{-(pu')' + qu\right\}\,dx \tag{30.25}$$

$$= -vpu'|_0^\ell + \int_0^\ell u'pv'\,dx + \int_0^\ell uqv\,dx \tag{30.26}$$

$$= -vpu'|_0^\ell + upv'|_0^\ell + \int_0^\ell u\left\{-(pv')' + qv\right\}dx \tag{30.27}$$

$$\text{Therefore } \int_0^\ell v\mathcal{L}u\,dx = -pvu'|_0^\ell + puv'|_0^\ell + \int_0^\ell u\mathcal{L}v\,dx. \qquad \square \tag{30.28}$$

Now suppose that u and v both satisfy the SL boundary conditions. I.E.

$$\begin{array}{rclrcl} \alpha_1 u(0) + \alpha_2 u'(0) & = & 0 & \beta_1 u(\ell) + \beta_2 u'(\ell) & = & 0 \\ \alpha_1 v(0) + \alpha_2 v'(0) & = & 0 & \beta_1 v(\ell) + \beta_2 v'(\ell) & = & 0 \end{array} \tag{30.29}$$

then

$$\int_0^\ell v\mathcal{L}u\,dx - \int_0^\ell u\mathcal{L}v\,dx = -p(\ell)u'(\ell)v(\ell) + p(\ell)u(\ell)v'(\ell) \tag{30.30}$$

$$+p(0)u'(0)v(0) - p(0)u(0)v'(0) \tag{30.31}$$

$$= p(\ell)\left\{+\frac{\beta_1}{\beta_2}u(\ell)v(\ell) + u(\ell)\left(-\frac{\beta_1}{\beta_2}v(\ell)\right)\right\} \tag{30.32}$$

$$+p(0)\left\{-\frac{\alpha_1}{\alpha_2}u(0)v(0) - u(0)\left(-\frac{\alpha_1}{\alpha_2}v(0)\right)\right\} \tag{30.33}$$

$$= 0. \tag{30.34}$$

Thus $\int_0^\ell v\mathcal{L}u\,dx = \int_0^\ell u\mathcal{L}v\,dx$ whenever u and v satisfy the SL boundary condition.

Observations:

- If $\mathcal{L}$ and BC are such that $\int_0^\ell v\mathcal{L}u\,dx = \int_0^\ell u\mathcal{L}v\,dx$ then $\mathcal{L}$ is said to be **self-adjoint**.
- Notation: if we define $(f,g) = \int_0^\ell f(x)g(x)\,dx$ then we may write $(v,\mathcal{L}v) = (u,\mathcal{L}v)$.

30.4.2 *Proofs using Lagrange's Identity:*

(1a) The λ_j are real: Let $\mathcal{L}y = \lambda ry$ (1) $\alpha_1 y(0) + \alpha_2 y'(0) = 0$ $\beta_1 y(\ell) + \beta_2 y'(\ell) = 0$. Take the conjugate of (1) $\mathcal{L}\bar{y} = \bar{\lambda}r\bar{y}$. By Lagrange's Identity:

$$
\begin{aligned}
0 &= (\bar{y},\mathcal{L}y) - (y,\mathcal{L}\bar{y}) && (30.35)\\
&= (\bar{y}, r\lambda y) - (y, r\bar{\lambda}\bar{y}) && (30.36)\\
&= \int_0^\ell \bar{y}(x)r\lambda y(x)\,dx - \int_0^\ell y(x)r(x)\bar{\lambda}\bar{y}(x)\,dx && (30.37)\\
&= (\lambda-\bar{\lambda})\int_0^\ell r(x)\big|y(x)\big|^2\,dx && (30.38)
\end{aligned}
$$

Since $r(x)|y(x)|^2 \geq 0$ it follows that $\lambda = \bar{\lambda} \Rightarrow \lambda$ is real.
(1c) $\lambda_j > 0$ provided $\alpha_1/\alpha_2 < 0$ $\beta_1/\beta_2 > 0$ and $q(x) > 0$. Consider $\mathcal{L}y = -(py')' + qy = \lambda ry$ (SL) and multiply (SL) by y and integrate from 0 to ℓ:

$$
(y,\mathcal{L}y) = \int_0^\ell -(py')'y + qy^2\,dx = \lambda\int_0^\ell r(x)\big[y(x)\big]^2\,dx \tag{30.39}
$$

Therefore $\lambda = \dfrac{\int_0^\ell -(py')'y + qy^2\,dx}{\int_0^\ell ry^2\,dx}$ this is known as Rayleigh's Quotient.

$$
= \frac{[-py'y]_0^\ell + \int_0^\ell p(y')^2 + qy^2\,dx}{\int_0^\ell ry^2\,dx} \tag{30.40}
$$

$$
= \frac{+p(\ell)\frac{\beta_1}{\beta_2}\big[y(\ell)\big]^2 - p(0)\frac{\alpha_1}{\alpha_2}\big[y(0)\big]^2 + \int_0^\ell p(y')^2 + qy^2\,dx}{\int_0^\ell ry^2\,dx}. \tag{30.41}
$$

Therefore $\lambda > 0$ since the RHS is all positive.
Note: If $q(x) \equiv 0$ and $\alpha_1 = 0 = \beta_1$ then with $y'(0) = 0 = y'(\ell)$ we have nontrivial eigenfunction $y(x) = 1$ and eigenvalue $\lambda = 0$.

(2b) Eigenfunctions corresponding to different eigenvalues are orthogonal. Consider two distinct eigenvalues $\lambda_j \neq \lambda_k$ $\lambda_j : \mathcal{L}\phi_j = r\lambda_j\phi_j$ and $\lambda_k : \mathcal{L}\phi_k = r\lambda_k\phi_k$. Then

$$0 = (\phi_k, \mathcal{L}\phi_j) - (\phi_j, \mathcal{L}\phi_k) \quad \text{by Lagrange's Identity} \tag{30.42}$$

$$= (\phi_k, r\lambda_j\phi_j) - (\phi_j, r\lambda_k\phi_k) \tag{30.43}$$

$$= (\lambda_j - \lambda_k)\int_0^\ell r(x)\phi_k(x)\phi_j(x)\,dx \tag{30.44}$$

now $\lambda_j \neq \lambda_k$ implies that

$$\int_0^\ell r(x)\phi_k(x)\phi_j(x)\,dx = 0. \tag{30.45}$$

(3) The eigenfunctions form a complete set: It is difficult to prove the convergence of the eigenfunction series expansion for $f(x)$ that is piecewise smooth. However, if we assume the expansion converges then it is a simple matter to use orthogonality to determine the coefficients in the expansion: Let $f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$.

$$\int_0^\ell f(x)\phi_m(x)r(x)\,dx = \sum_{n=1}^{\infty} c_n \int_0^\ell r(x)\phi_m(x)\phi_n(x)\,dx \tag{30.46}$$

orthogonality implies

$$c_m = \frac{\int_0^\ell r(x)f(x)\phi_m(x)\,dx}{\int_0^\ell r(x)\big[\phi_m(x)\big]^2\,dx}. \tag{30.47}$$