Lecture 14: Half Range Fourier Series: even and odd functions

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In this lecture we consider the Fourier Expansions for Even and Odd functions, which give rise to cosine and sine half range Fourier Expansions. If we are only given values of a function $f(x)$ over half of the range $[0, L]$, we can define two different extensions of f to the full range $[-L, L]$, which yield distinct Fourier Expansions. The even extension gives rise to a half range cosine series, while the odd extension gives rise to a half range sine series.

Key Concepts: Even and Odd Functions; Half Range Fourier Expansions; Even and Odd Extensions

14.1 Even and Odd Functions

Even: $f(-x) = f(x)$

Odd: $f(-x) = -f(x)$

14.1.1 *Integrals of Even and Odd Functions*

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx \tag{14.1}$$

$$= \int_{0}^{L} \big[f(-x) + f(x)\big]\,dx \tag{14.2}$$

$$= \begin{cases} 2\int_0^L f(x)\,dx & f \text{ even} \\ 0 & f \text{ odd.} \end{cases} \tag{14.3}$$

Notes: Let $E(x)$ represent an even function and $O(x)$ an odd function.

(1) If $f(x) = E(x)\cdot O(x)$ then $f(-x) = E(-x)O(-x) = -E(x)O(x) = -f(x) \Rightarrow f$ is odd.

(2) $E_1(x)\cdot E_2(x) \to$ even.

(3) $O_1(x)\cdot O_2(x) \to$ even.

(4) Any function can be expressed as a sum of an even part and an odd part:

$$f(x) = \frac{1}{2}\underbrace{\big[f(x) + f(-x)\big]}_{\text{even part}} + \frac{1}{2}\underbrace{\big[f(x) - f(-x)\big]}_{\text{odd part}}. \tag{14.4}$$

Check: Let $E(x) = \frac{1}{2}\big[f(x) + f(-x)\big]$. Then $E(-x) = \frac{1}{2}\big[f(-x) + f(x)\big] = E(x)$ even. Similarly let

$$O(x) = \frac{1}{2}\big[f(x) - f(-x)\big] \tag{14.5}$$

$$O(-x) = \frac{1}{2}\big[f(-x) - f(x)\big] = -O(x) \text{ odd.} \tag{14.6}$$

14.2 Consequences of the Even/Odd Property for Fourier Series

(I) Let $f(x)$ be Even-Cosine Series:

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos}_{\text{even}}\left(\frac{n\pi x}{L}\right)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \tag{14.7}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\left(\frac{n\pi x}{L}\right)}_{\text{odd}}\,dx = 0. \tag{14.8}$$

Therefore

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right); \quad a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx. \tag{14.9}$$

(II) Let $f(x)$ be Odd-Sine Series:

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\left(\frac{n\pi x}{L}\right)}_{\text{odd}}\,dx = 0 \tag{14.10}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\left(\frac{n\pi x}{L}\right)}_{\text{even}}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

Therefore

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right); \quad b_n = \frac{2}{L}\int_{-0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

(III) Since any function can be written as the sum of an even and odd part, we can interpret the cos and sin series as even/odd:

$$\begin{aligned} f(x) = {} & \overset{\text{even}}{\frac{1}{2}\big[f(x) + f(-x)\big]} \quad + \overset{\text{odd}}{\frac{1}{2}\big[f(x) - f(-x)\big]} \\ = {} & \left\{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)\right\} + \left\{\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\right\} \end{aligned} \tag{14.11}$$

where

$$a_n = \frac{2}{L}\int_0^L \frac{1}{2}\big[f(x)+f(-x)\big]\cos\left(\frac{n\pi x}{L}\right)\,dx = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

$$b_n = \frac{2}{L}\int_0^L \frac{1}{2}\big[f(x)-f(-x)\big]\sin\left(\frac{n\pi x}{L}\right)\,dx = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

14.3 Half-Range Expansions

If we are given a function $f(x)$ on an interval $[0,L]$ and we want to represent f by a Fourier Series we have two choices - a Cosine Series or a Sine Series.
Cosine Series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \tag{14.12}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx. \tag{14.13}$$

Sine Series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{14.14}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx. \tag{14.15}$$

Example 14.1 *Expand $f(x) = x$, $0 < x < 2$ in a half-range (a) Sine Series, (b) Cosine Series.*

(a) Sine Series: (L=2)

$$b_n = \frac{2}{L}\int_0^L f(t)\sin\frac{n\pi}{\ell}t\,dt \tag{14.16}$$

$$= \int_0^2 t\sin\frac{n\pi}{2}t\,dt \tag{14.17}$$

$$= -\left.\frac{t\cos\frac{n\pi}{2}t}{\left(\frac{n\pi}{2}\right)}\right|_0^2 + \frac{2}{n\pi}\int_0^2 \cos\frac{n\pi}{2}t\,dt \tag{14.18}$$

$$= -\frac{4}{n\pi}\cos(n\pi) + \left(\frac{2}{n\pi}\right)^2 \sin\left.\left(\frac{n\pi}{2}t\right)\right|_0^2 \tag{14.19}$$

$$= -\frac{4}{n\pi}(-1)^n \tag{14.20}$$

Therefore

$$f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{2}t\right). \tag{14.21}$$

$$f(1) = 1 = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi}{2}\right) \tag{14.22}$$

$$\textit{therefore} \quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \tag{14.23}$$

(b) Cosine Series: (L=2)

$$a_0 = \frac{2}{2} \int_0^2 t\,dt = \left.\frac{t^2}{2}\right|_0^2 = 2 \tag{14.24}$$

$$a_n = \int_0^2 t \cos \frac{n\pi}{2} t\,dt = \left(\frac{2}{n\pi}\right) t \sin \left.\frac{n\pi}{2} t\right|_0^2 - \left(\frac{2}{n\pi}\right) \int_0^2 \sin \frac{n\pi}{2} t\,dt$$

$$= + \left(\frac{2}{n\pi}\right)^2 \cos \left.\frac{n\pi}{2} t\right|_0^2 = \frac{4}{n^2\pi^2} \{\cos n\pi - 1\} \tag{14.25}$$

Therefore

$$f(t) = 1 + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\left[(-1)^n - 1\right]}{n^2} \cos \frac{n\pi}{2} t \tag{14.26}$$

$$= 1 - \frac{8}{\pi^2} \sum_{n=0}^{\infty} \cos \frac{(2n+1)}{2} \pi t / (2n+1)^2. \tag{14.27}$$

The cosine series converges faster than Sine Series.

$$f(2) = 2 = 1 + \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \qquad \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Example 14.2 *Periodic Extension: Assume that $f(x) = x$, $0 < x < 2$ represents one full period of the function so that $f(x+2) = f(x)$. $2L = 2 \Rightarrow L = 1$.*

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\,dx = \int_{-1}^{1} f(x)\,dx = \int_0^2 x\,dx = \left.\frac{x^2}{2}\right|_0^2 = 2 \tag{14.28}$$

$$\textit{since } f(x+2) = f(2). \tag{14.29}$$

$n \geq 1$:

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx \quad L=1 \\
&= \int_{0}^{2} x\cos(n\pi x)\,dx \\
&= \left[\frac{x\sin(n\pi x)}{n\pi}\bigg|_{0}^{2} - \left(\frac{1}{n\pi}\right)\int_{0}^{2}\sin(n\pi x)\,dx\right] \\
&= \frac{1}{(n\pi)^2}\cos(n\pi x)\bigg|_{0}^{2} = \frac{1}{(n\pi)^2}\big[\cos(2n\pi)-1\big] = 0 \qquad (14.30) \\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx \\
&= \int_{0}^{2} x\sin(n\pi x)\,dx = \left[-x\frac{\cos(n\pi x)}{n\pi}\bigg|_{0}^{2} + \frac{1}{(n\pi)}\int_{0}^{2}\cos(n\pi x)\,dx\right] \\
&= \frac{-2}{n\pi} + \frac{\sin(n\pi x)}{(n\pi)^2}\bigg|_{0}^{2} = \left(\frac{-2}{n\pi}\right) \qquad (14.31)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f(x) &= \frac{2}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi x)}{n} \qquad (14.32) \\
&= 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi x)}{n}
\end{aligned}
$$

(14.33)

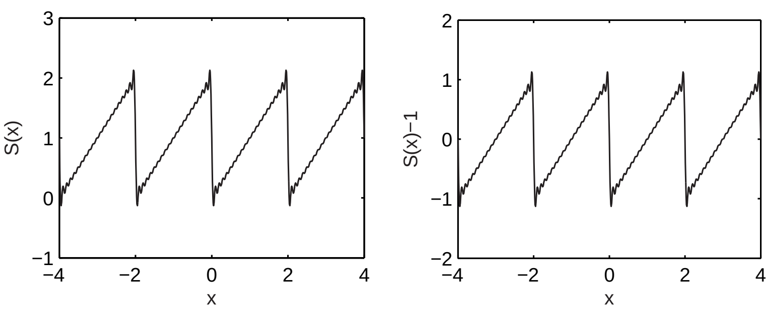

FIGURE 1. Left figure: Full Range Expansion $S_N(x) = 1 - \frac{2}{\pi}\sum_{n=1}^{N=20}\frac{\sin(n\pi x)}{n}$ Right figure: An odd function $S_N(x) - 1 = -\frac{2}{\pi}\sum_{n=1}^{N=20}\frac{\sin(n\pi x)}{n}$