Lecture 1: Review of methods to solve Ordinary Differential Equations

(*Compiled 3 September 2014*)

In this lecture we will briefly review some of the techniques for solving First Order ODE and Second Order Linear ODE, including Cauchy-Euler/Equidimensional Equations

Key Concepts: First order ODEs: Separable and Linear equations; Second Order Linear ODEs: Constant Coefficient Linear ODE, Cauchy-Euler/Equidimensional Equations.

1 First Order ordinary Differential Equations:

1.1 Separable Equations:

$$\begin{aligned}\frac{dy}{dx} &= P(x)Q(y) \qquad (1.1)\\ \int \frac{dy}{Q(y)} &= \int P(x)\,dx + C\end{aligned}$$

Example 1:

$$\begin{aligned}\frac{dy}{dx} &= \frac{4y}{x(y-3)}\\ \left(\frac{y-3}{y}\right)dy &= \frac{4}{x}\,dx\\ y - 3\ln|y| &= 4\ln|x| + C \qquad (1.2)\\ y &= \ln(x^4y^3) + C\\ Ax^4y^3 &= \mathrm{e}^y\end{aligned}$$

1.2 Linear First Order equations - The Integrating Factor:

$$y'(x) + P(x)y = Q(x) \qquad (1.3)$$

Can we find a function $F(x)$ to multiply (4.3) by in order to turn the left hand side into a derivative of a product:

$$Fy' + FPy = FQ \qquad (1.4)$$

$$(Fy)' = Fy' + F'y = FQ \qquad (1.5)$$

So let $F' = FP$ which is a separable Eq.

$$\frac{dF}{F(x)} = P(x)\,dx \Rightarrow \int \frac{dF}{F} = \int P(x)\,dx + C$$

$$\text{Therefore } \ln F = \int P(x)\,dx + C \tag{1.6}$$

$$\text{or } F = A\mathrm{e}^{\int P(x)\,dx} \quad \text{choose } A = 1$$

$$F = \mathrm{e}^{\int P(x)\,dx} \quad \text{integrating factor}$$

Therefore

$$\begin{aligned} \mathrm{e}^{\int P(x)\,dx} y' + \mathrm{e}^{\int P(x)\,dx} P(x) y &= \mathrm{e}^{\int P(x)\,dx} Q(x) \\ \left(\mathrm{e}^{\int P(x)\,dx} y\right)' &= \mathrm{e}^{\int P(x)\,dx} Q(x) \\ y(x) &= \mathrm{e}^{-\int P(x)\,dx} \left\{ \int \mathrm{e}^{\int^x P(t)\,dt} Q(x)\,dx + C \right\} \end{aligned} \tag{1.7}$$

Example 2:

$$y' + 2y = 0 \tag{1.8}$$

$$\begin{aligned} F(x) = \mathrm{e}^{2x} \Rightarrow \mathrm{e}^{2x} y' + \mathrm{e}^{2x} 2y &= (\mathrm{e}^{2x} y)' = 0 \\ \mathrm{e}^{2x} y &= c \\ y(x) &= C\mathrm{e}^{-2x} \end{aligned}$$

Example 3: Solve

$$\frac{dy}{dx} + \cot(x) y = 5\mathrm{e}^{\cos x},\ y(\pi/2) = -4 \tag{1.9}$$

$$\begin{array}{rclcrcl} P(x) & = & \cot x & Q(x) & = & 5\mathrm{e}^{\cos x} & \\ F(x) & = & \mathrm{e}^{\int \cot x\,dx} & & = & \mathrm{e}^{\ln(\sin x)} & = \sin x \end{array} \tag{1.10}$$

$$\text{Therefore } \sin(x) y' + \cos(x) y = (\sin(x) y)' = 5\mathrm{e}^{\cos x} \sin x$$

$$\sin(x) y = -5\mathrm{e}^{\cos x} + C$$

$$y(x) = -\tfrac{5\mathrm{e}^{\cos x} - C}{\sin x} \tag{1.11}$$

$$-4 = y(\pi/2) = -\tfrac{5-C}{1} \Rightarrow C = 1$$

$$\text{Therefore } y(x) = \tfrac{1 - 5\mathrm{e}^{\cos x}}{\sin x}$$

2 Second Order Constant Coefficient Linear Equations:

$$\begin{aligned} Ly = ay'' + by' + cy &= 0 \\ \text{Guess } y = \mathrm{e}^{rx} \quad y' = r\mathrm{e}^{rx} &\quad y'' = r^2 \mathrm{e}^{rx} \\ Ly = [ar^2 + br + c]\mathrm{e}^{rx} &= 0 \text{ provided } [] = 0 \end{aligned}$$

Indicial Eq.:

$$\begin{array}{rcll} g(r) = ar^2 + br + c & = & 0 & \quad r_{1,2} = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a} \\ \text{or } g(r) = a(r - r_1)(r - r_2) & = & 0 & \end{array} \tag{2.1}$$

Case I: $\Delta = b^2 - 4ac > 0, r_1 \neq r_2, y(x) = c_1 \mathrm{e}^{r_1 x} + c_2 \mathrm{e}^{r_2 x}$ is the general solution.

Case II: $\Delta = 0$, $r_1 = r_2$, repeated roots $Ly = a(r - r_1)^2 \mathrm{e}^{rx} = 0$. In this case obtain only *one* solution $y(x) = \mathrm{e}^{r_1 x}$. How do we get a second solution?

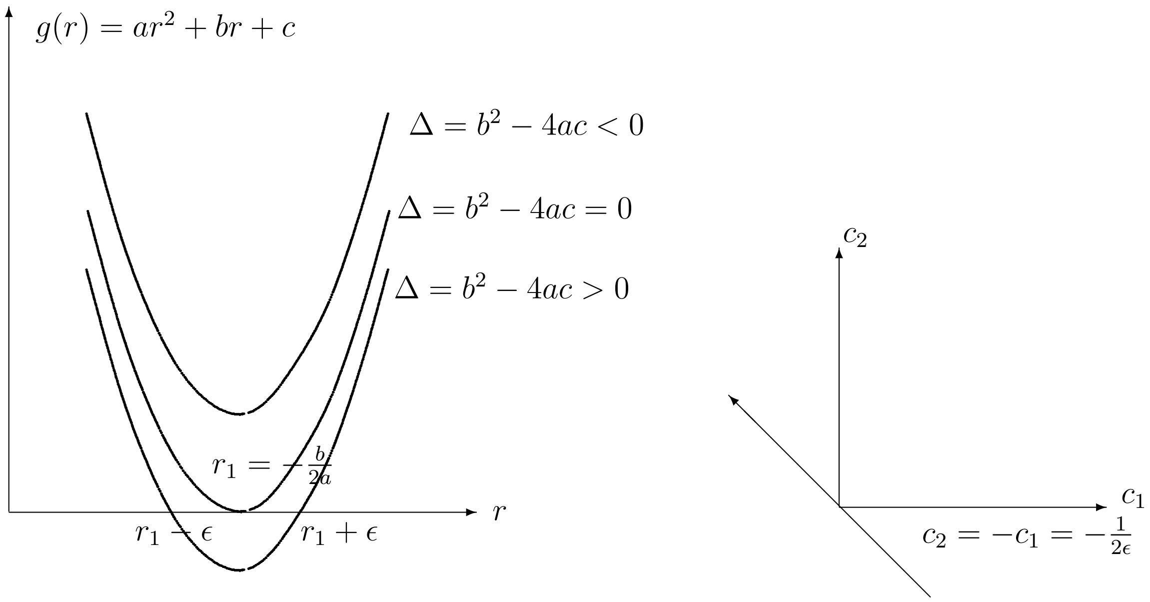

FIGURE 1. Left Figure: Roots of the characteristic polynomial $g(r) = ar^2 + br + c$ for the different cases of the discriminant $\Delta = b^2 - 4ac$. We consider special solution, in which $g(r) = a(r - (r_1 - \epsilon))(r - (r + \epsilon)) = a[(r - r_1)^2 - \epsilon^2] \approx a(r - r_1)^2$. Right Figure: We consider the special solution (2.3) for the case in which the two parameters c_1 and c_2 have been chosen to be $c_2 = -c_1 = -\frac{1}{2\epsilon}$, which represents a straight line in the two-parameter $c_1 - c_2$ space

First Method: Perturbation of the double root: Consider a small perturbation (see figure 1 a) to the double root case, such that $g(r) = a(r - (r_1 - \epsilon))(r - (r_1 + \epsilon)) = a[(r - r_1)^2 - \epsilon^2] \approx a(r - r_1)^2$. In this case the two, very close but distinct, roots of $g(r) = 0$ are given by:

$$r = r_1 + \epsilon \text{ and } r = r_1 - \epsilon \tag{2.2}$$

Now since we still have two distinct roots in this perturbed case, the general solution is:

$$y(x) = c_1 \mathrm{e}^{(r_1+\epsilon)x} + c_2 \mathrm{e}^{(r_1-\epsilon)x} \tag{2.3}$$

Now choosing a special solution by selecting $c_1 = \frac{1}{2\epsilon} = -c_2$, and we obtain a family of solutions that depend on the small parameter ϵ (see figure 1 b):

$$y(x, \epsilon) = \frac{\mathrm{e}^{(r_1+\epsilon)x} - \mathrm{e}^{(r_1-\epsilon)x}}{2\epsilon} \approx \left| \frac{\partial}{\partial r} \mathrm{e}^{rx} \right|_{r=r_1} \tag{2.4}$$

Now taking the limit as $\epsilon \to 0$ by making use of L'Hospital's Rule, we obtain the following limiting solution:

$$y(x, \epsilon) = \mathrm{e}^{r_1 x} \left(\frac{e^{\epsilon x} - \mathrm{e}^{-\epsilon x}}{2\epsilon} \right) \xrightarrow{\epsilon \to 0} x\mathrm{e}^{r_1 x} = \left| \frac{\partial}{\partial r} \mathrm{e}^{rx} \right|_{r=r_1} \tag{2.5}$$

Second Method: taking the derivative with respect to r: From (2.4) and (2.5) we see that the new solution $x\mathrm{e}^{r_1 x}$ was obtained by taking the derivative of $y(x, r) = \mathrm{e}^{rx}$ with respect to r and then making the substitution $r = r_1$. This is, in fact, a general procedure that we will use later in the course. To see why this procedure works, let

$$\begin{aligned} y(r,x) &= \mathrm{e}^{rx} \\ Ly(r,x) &= a(r-r_1)^2\mathrm{e}^{rx} \\ L\left[\frac{\partial y}{\partial r}(r,x)\right]_{r=r_1} &= \left[2a(r-r_1)\mathrm{e}^{rx} + 2a(r-r_1)x\mathrm{e}^{rx}\right]_{r=r_1} = 0 \\ \text{Therefore } \left[\frac{\partial y}{\partial r}(r,x)\right]_{r=r_1} &= x\mathrm{e}^{r_1 x} \text{ is also a solution.} \end{aligned} \tag{2.6}$$

Thus, to summarize, the general solution for the case of a double root is:

$$y(x) = c_1\mathrm{e}^{r_1 x} + c_2 x\mathrm{e}^{r_1 x} \tag{2.7}$$

Case III: Complex Conjugate Roots: $\Delta = b^2 - 4ac < 0$

$$\begin{aligned} r_\pm &= -\frac{b}{2a} \pm i\left[4ac - b^2\right]^{1/2} = \lambda \pm i\mu \\ y(x) &= c_1\mathrm{e}^{(\lambda+i\mu)x} + c_2\mathrm{e}^{(\lambda-i\mu)x} \\ &= \mathrm{e}^{\lambda x}\left[A\cos\mu x + B\sin\mu x\right]. \end{aligned} \tag{2.8}$$

Example 4:

$$\begin{aligned} Ly &= y'' + y' - 6y = 0 \\ y &= \mathrm{e}^{rx}(r^2 + r - 6) = (r+3)(r-2) = 0 \\ y(x) &= c_1\mathrm{e}^{-3x} + c_2\mathrm{e}^{2x} \end{aligned} \tag{2.9}$$

Example 5:

$$\begin{aligned} Ly &= y'' + 6y' + 9y = 0 \\ y &= \mathrm{e}^{rx}(r+3)^2 = 0 \\ y(x) &= c_1\mathrm{e}^{-3x} + c_2 x\mathrm{e}^{-3x} \end{aligned} \tag{2.10}$$

Example 6:

$$\begin{aligned} Ly &= y'' - 4y' + 13y = 0 \\ y &= \mathrm{e}^{rx}: \quad r^2 - 4r + 13 = 0 \\ & \qquad\quad r = \tfrac{4\pm\sqrt{16-52}}{2} = 2 \pm 3i \\ \text{Therefore } y(x) &= \mathrm{e}^{2x}\left[A\cos 3x + B\sin 3x\right]. \end{aligned} \tag{2.11}$$

3 Cauchy/Euler/Equidimensional Equations:

$$Ly = x^2 y'' + \alpha x y' + \beta y = 0. \tag{3.1}$$

Aside: Note if we let $t = \ln x$ or $x = \mathrm{e}^t$ then $\dfrac{d}{dx} = \dfrac{d}{dt}\dfrac{dt}{dx} \Rightarrow \dfrac{d}{dt} = x\dfrac{d}{dx}$.

$$\frac{d^2}{dt^2} = x\frac{d}{dx}\left(x\frac{d}{dx}\right) = x^2\frac{d^2}{dx^2} + x\frac{d}{dx} \Rightarrow x^2\frac{d^2}{dx^2} = \frac{d^2}{dt^2} - \frac{d}{dt} \tag{3.2}$$

$$\begin{aligned} \text{Therefore } \quad \ddot{y} - \dot{y} + \alpha\dot{y} + \beta y &= 0 \\ \ddot{y} + (\alpha - 1)\dot{y} + \beta y &= 0 \end{aligned} \tag{3.3}$$

$$y = \mathrm{e}^{rt} \Rightarrow r^2 + (\alpha-1)r + \beta = 0 \quad \text{Characteristic Eq.}$$

Back to (3.1): Guess $y = x^r$, $y' = rx^{r-1}$, and $y'' = r(r-1)x^{r-2}$.

$$\begin{aligned} \text{Therefore} \quad \{r(r-1) + \alpha r + \beta\} x^r &= 0 \\ f(r) = r^2 + (\alpha - 1)r + \beta &= 0 \qquad \text{as above.} \end{aligned} \tag{3.4}$$

$$r_\pm = \frac{1 - \alpha \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2} \tag{3.5}$$

Case 1: $\Delta = (\alpha-1)^2 - 4\beta > 0$ Two Distinct Real Roots r_1, r_2.

$$y = c_1 x^{r_1} + c_2 x^{r_2} \tag{3.6}$$

If r_1 or $r_2 < 0$ then $|y| \to \infty$ as $x \to 0$.

Case 2: $\Delta = 0$ Double Root $(r - r_1)^2 = 0$.

We obtain only one solution in this case:

$$y \quad = \quad c_1 x^{r_1} \tag{3.7}$$

To get a second solution we use second method introduced above, in which we differentiate with respect to the parameter r:

$$\begin{aligned} \frac{\partial}{\partial r} L[x^r] &= L\left[\frac{\partial}{\partial r} x^r\right] = L[x^r \log x] \\ \frac{\partial}{\partial r}\{f(r) x^r\} &= f'(r) x^r + f(r) x^r \log x = 0 \qquad \text{since } f(r) = (r - r_1)^2. \end{aligned} \tag{3.8}$$

General Solution: $y(x) = (c_1 + c_2 \log x) x^{r_1}$.

Check:

$$\begin{aligned} L(x^{r_1} \log x) &= x^2 (x^r \log x)'' + \alpha x (x^r \log x)' + \beta (x^r \log x) - \\ &= x^2 \left[r(r-1) x^r \log x + r x^{r-2} + (r-1) x^{r-2}\right] \\ &\quad + \alpha x \left[r x^{r-1} \log x + x^{r-1}\right] + \beta (x^r \log x) \\ &= \left\{r^2 + (\alpha - 1) r + \beta\right\} x^r \log x + \{2r - 1 + \alpha\} x^r = 0 \end{aligned} \tag{3.9}$$

Case 3: $\Delta = (\alpha - 1)^2 - 4\beta < 0$.

$$\begin{aligned} r_\pm &= \frac{(1-\alpha)}{2} \pm i \frac{\left[4\beta - (\alpha-1)^2\right]^{1/2}}{2} = \lambda \pm i\mu \\ y(x) &= c_1 x^{(\lambda + i\mu)} + c_2 x^{(\lambda - i\mu)} \qquad x^r = \mathrm{e}^{r \ln x} \\ &= c_1 \mathrm{e}^{(\lambda + i\mu) \ln x} + c_2 \mathrm{e}^{(\lambda - i\mu) \ln x} \\ &= x^\lambda \left\{c_1 \mathrm{e}^{i\mu \ln x} + c_2 \mathrm{e}^{-i\mu \ln x}\right\} \\ &= A_1 x^\lambda \cos(\mu \ln x) + A_2 x^\lambda \sin(\mu \ln x) \end{aligned} \tag{3.10}$$

Observations:

- If $x < 0$ replace by $|x|$.

- The two solutions are linearly independent as we can verify by applying the Wronskian test, as follows:

$$
\begin{aligned}
w(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 \quad \text{(look up the definition of the Wronskian)} \\
&= \left\{x^\lambda \cos(\mu \ln x)\right\} \left\{\log x x^\lambda \sin(\mu \ln x) + x^{\lambda-1} \cos(\mu \ln x)\mu\right\} \\
&\quad - \left\{x^\lambda \log x \cos(\mu \ln x) - x^{\lambda-1} \sin(\mu \ln x)\mu\right\} \left\{x^\lambda \sin(\mu \ln x)\right\} \\
&= \mu x^{2\lambda-1} \quad \text{independent for } x \neq 0.
\end{aligned}
$$

Example 7:

$$
\begin{gathered}
x^2 y'' - x y' - 2y = 0, \quad y(1) = 0, \quad y'(1) = 1 \\
y = x^r \quad r(r-1) - r - 2 = 0 \quad r^2 - 2r - 2 = 0 \\
(r-1)^2 = 3 \quad r = 1 \pm \sqrt{3}
\end{gathered} \tag{3.11}
$$

$$
\begin{aligned}
y &= c_1 x^{1+\sqrt{3}} + c_2 x^{1-\sqrt{3}} \\
y(1) &= c_1 + c_2 = 0 \quad c_2 = -c_1 \\
y(x) &= c_1 \left(x^{1+\sqrt{3}} - x^{1-\sqrt{3}}\right) \\
y'(x) &= c_1 \left[(1+\sqrt{3}) x^{\sqrt{3}} - (1-\sqrt{3}) x^{-\sqrt{3}}\right]\Big|_{x=1} = c_1 2\sqrt{3} = 1
\end{aligned} \tag{3.12}
$$

$$
\text{Therefore} \quad y(x) = \frac{1}{2\sqrt{3}} \left(x^{1+\sqrt{3}} - x^{1-\sqrt{3}}\right). \tag{3.13}
$$

Example 8:

$$
\begin{gathered}
x^2 y'' - 3xy' + 4y = 0 \quad y(1) = 1 \quad y'(1) = 0 \\
y = x^r \implies r(r-1) - 3r + 4 = r^2 - 4r + 4 = 0 \quad (r-2)^2 = 0
\end{gathered} \tag{3.14}
$$

$$
\begin{aligned}
y(x) &= c_1 x^2 + c_2 x^2 \log x \\
y(1) &= c_1 = 1 \quad y'(x) = 2x + c_2 \left[2x \log x + x\right]_{\alpha=1} \\
&\qquad = 2 + c_2 = 0
\end{aligned} \tag{3.15}
$$

$$
\text{Therefore} \quad y(x) = x^2 - 2x^2 \log x.
$$