M316/257 HW 3

PROBLEM 1 FIND SERIES EXPANSIONS FOR TWO LINEARLY INDEPENDENT SOLUTIONS OF

$$y'' + \frac{(1-x)}{x} y' - \frac{y}{x^2} = 0 \quad \text{ABOUT } x = 0.$$

SHOW THAT ALTHOUGH THE ROOTS OF THE INDICIAL EQUATION DIFFER BY AN INTEGER, WE CAN NEVERTHELESS FIND TWO FROBENIUS SERIES SOLUTIONS.

PROBLEM 2 CONSIDER BESSEL'S EQUATION OF ORDER $1/2$ GIVEN BY

$$(*) \quad x^2 y'' + x y' + (x^2 - 1/4) y = 0.$$

(i) FIND FROBENIUS SERIES EXPANSIONS FOR TWO LINEARLY INDEPENDENT SOLUTIONS OF THE FORM $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. SHOW THAT ALTHOUGH THE ROOTS OF THE INDICIAL EQUATION DIFFER BY AN INTEGER WE CAN FIND y_1, y_2.

(ii) SHOW HOW TO FIND THE SOLUTIONS y_1 AND y_2 BY SUBSTITUTING $y = x^{-1/2} v$ INTO $(*)$ AND DERIVING THE EQUATION FOR v.

PROBLEM 3 FIND THE FIRST THREE NON-ZERO TERMS IN A FROBENIUS SERIES SOLUTION ABOUT $x = 0$ FOR THE SOLUTION TO THE SPHERICAL BESSEL'S EQUATION

$$x^2 y'' + 2x y' + (x^2 - 5/16) y = 0$$

WITH $\quad y(0) = 0, \quad \lim_{x \to 0^+} x^{3/4} y'(x) = 1.$

(HINT: USE THE INITIAL VALUES TO IDENTIFY WHICH OF THE TWO LINEARLY INDEPENDENT SOLUTIONS IS NEEDED).

PROBLEM 4 FIND THE FORMS OF TWO LINEARLY INDEPENDENT FROBENIUS SERIES EXPANSIONS ABOUT THE POINT $X=0$ FOR

$$y'' + \frac{1}{\sin x} y' + \frac{(1-x)}{x^2} y = 0$$

SUCH THAT THE SERIES ARE REAL-VALUED ON THE POSITIVE REAL AXIS $X>0$. DO NOT CALCULATE THE COEFFICIENTS IN THESE SERIES EXPANSIONS.

PROBLEM 5 IN THE NOTES ONE SOLUTION OF BESSEL'S EQUATION OF ORDER ZERO

$$x^2 y'' + x y' + x^2 y = 0$$

IS FOUND TO BE

$$J_0(x) = 1 - \frac{1}{2^2} x^2 + \frac{1}{2^2 4^2} x^4 - \frac{1}{2^2 4^2 6^2} x^6 + \dots = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m} (m!)^2} x^{2m}.$$

FIND A SECOND LINEARLY INDEPENDENT SOLUTION OF THE FORM

$$y_2 = J_0(x) \ln x + \tilde{y}(x) \quad \text{AS FOLLOWS:}$$

(i) SHOW THAT IF y_2 IS TO SOLVE BESSEL'S EQUATION THEN $\tilde{y}$ MUST SOLVE

$$(*) \quad x^2 \tilde{y}'' + x \tilde{y}' + x^2 \tilde{y} = -2x J_0'(x).$$

(ii) FIND THE FIRST THREE NON-ZERO TERMS OF A SERIES EXPANSION $\tilde{y} = \sum_{n=1}^{\infty} b_n x^n$ FOR THIS ODE FOR $\tilde{y}$.

(HINT: PUT SERIES FOR $J_0(x)$ INTO THE RIGHT-HAND SIDE OF $(*)$).

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PROBLEM 1 $\quad y'' + \frac{(1-x)}{x} y' - \frac{y}{x^2} = 0.$

MULTIPLY BY x^2: $\quad x^2 y'' + [x - x^2] y' - y = 0.$ $\quad x = 0$ IS A REG. SING. POINT

NOW PUT $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. THEN

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r+1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

THIS YIELDS THAT $\quad \sum_{n=0}^{\infty} \left(a_n (n+r)^2 x^{n+r} - a_n \right) - \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r} = 0$

NOW $\quad x^r a_0 [r^2 - 1] + \sum_{n=1}^{\infty} x^{n+r} \left[a_n \left((n+r)^2 - 1 \right) - (n+r-1) a_{n-1} \right] = 0.$

THUS $r = \pm 1$ ARE ROOTS OF INDICIAL EQUATION AND

$$a_n = \frac{(n+r-1) a_{n-1}}{(n+r)^2 - 1}, \quad n = 1, 2, \ldots \qquad \begin{aligned} r_1 &= 1 \\ r_2 &= -1 \\ r_1 - r_2 &= 2. \end{aligned}$$

LARGEST ROOT $(r_1 = 1)$. WE HAVE

$$a_n = \frac{n \, a_{n-1}}{n^2 + 2n} = \frac{a_{n-1}}{n+2}.$$

THUS $\quad a_1 = \frac{a_0}{3}, \quad a_2 = \frac{a_1}{4} = \frac{a_0}{3 \cdot 4}, \quad a_3 = \frac{a_2}{5} = \frac{a_0}{3 \cdot 4 \cdot 5}$

THUS $\quad a_n = \frac{2 a_0}{(n+2)!}, \quad n = 0, 1, 2, \ldots$

SO $\quad y_1(x) = x \sum_{n=0}^{\infty} \frac{2 a_0}{(n+2)!} x^n = \frac{2 a_0}{x} \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!}.$

OR $$y_1(x) = \frac{2a_0}{x} \sum_{n=2}^{\infty} \frac{x^n}{n!} = \frac{2a_0}{x}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} - 1 - x \right)$$

WE IDENTIFY $$y_1(x) = \frac{2a_0}{x}\left[e^x - 1 - x \right].$$

<u>SMALLEST ROOT</u> $(r_2 = -1)$. THEN,

$$a_n = \frac{(n-2)\, a_{n-1}}{n^2 - 2n} = \frac{a_{n-1}}{n} \qquad n = 1, 2, \ldots$$

THUS $$a_n = \frac{a_0}{n!} \quad \text{AND} \quad y_2(x) = x^{-1} \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n.$$

SO $$y_2(x) = \frac{a_0}{x} e^x.$$

WE HAVE FOUND y_2 ALTHOUGH $r_1 - r_2 =$ positive integer.

THUS $$y = \frac{c_1}{x}\left(e^x - 1 - x \right) + \frac{c_2}{x} e^x$$ ARE general solution.

NOTICE THAT THIS IS (BY LINEAR COMBINATION)

$$y = \frac{d_1}{x} e^x + d_2 \left(1 + \frac{1}{x} \right)$$

d_1, d_2 ARBITRARY.

PROBLEM 2

$$x^2 y'' + x y' + (x^2 - 1/4) y = 0.$$

(i) WE PUT $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. THEN

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - \frac{1}{4} \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

SHIFTING INDICES ON LAST TERM:

$$\sum_{n=0}^{\infty} (n+r)[(n+r-1)+1] a_n x^{n+r} - \frac{1}{4} \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0.$$

NOW EXTRACTING $n=0, 1$ WE GET

$$x^r \left(r^2 - \frac{1}{4}\right) a_0 + \left[(r+1)^2 - \frac{1}{4}\right] a_1 x^{r+1} + \sum_{n=2}^{\infty} \left(\left[(n+r)^2 - \frac{1}{4}\right] a_n + a_{n-2}\right) x^{n+r} = 0.$$

THUS IF a_0 ARBITRARY $\neq 0$ THEN $r = \pm 1/2$ ARE ROOTS OF INDICIAL EQUATION. THEN, $a_1 = 0$.

THUS $$a_n = \frac{-a_{n-2}}{[(n+r)^2 - 1/4]}, \quad n = 2, 3, \ldots$$

<u>TAKE ROOT $r_1 = 1/2$</u> : THEN $a_1 = a_3 = a_5 = \ldots = 0$.

WE HAVE $$a_n = \frac{-a_{n-2}}{n^2 + n} = -\frac{a_{n-2}}{n(n+1)}.$$

WE GET $$a_2 = -\frac{a_0}{2 \cdot 3}, \quad a_4 = -\frac{a_2}{4 \cdot 5} = \frac{a_0}{2 \cdot 3 \cdot 4 \cdot 5}, \quad a_6 = \frac{-a_0}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}$$

THUS $$a_{2n} = \frac{(-1)^n a_0}{(2n+1)!} \quad \text{AND} \quad n = 0, 1, 2, \ldots$$

THIS GIVES $$y_1(x) = x^{1/2} \sum_{n=0}^{\infty} a_{2n} x^{2n} = a_0 x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n}.$$

NOTICE THAT WE CAN WRITE

$$y_1(x) = a_0 x^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{AND IDENTIFY} \quad y_1(x) = a_0 x^{-1/2} \sin x.$$

<u>TAKE ROOT $r_2 = -1/2$</u> THEN $a_1 = a_3 = a_5 = \dots = 0$.

WE HAVE $$a_n = -\frac{a_{n-2}}{n(n-1)}, \quad n = 2, \dots$$

SO $$a_2 = -\frac{a_0}{1\cdot 2}, \quad a_4 = -\frac{a_2}{3\cdot 4} = \frac{a_0}{1\cdot 2\cdot 3\cdot 4}, \quad a_6 = -\frac{a_0}{1\cdot 2\cdots 6}.$$

THUS $$a_{2n} = \frac{(-1)^n a_0}{(2n)!}, \quad n = 0, 1, 2, \dots \qquad a_{2n+1} = 0, \quad n = 0, 1, 2, \dots$$

THIS GIVES $$y_2 = x^{-1/2} a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

WHICH WE IDENTIFY AS $y_2(x) = x^{-1/2} a_0 \cos x$.

AS SUCH THE GENERAL SOLUTION IS $y(x) = C_1 x^{-1/2} \cos x + C_2 x^{-1/2} \sin x$.
NO TROUBLE EVENTHOUGH $r_1 - r_2 =$ positive integer.

(iii) NOW PUT $y = x^{-1/2} v$ IN $x^2 y'' + x y' + (x^2 - 1/4) y = 0$.

WE CALCULATE $y' = -\frac{1}{2} x^{-3/2} v + x^{-1/2} v'$, $y'' = x^{-1/2} v'' + \frac{3}{4} x^{-5/2} v - x^{-3/2} v'$

SUBSTITUTE TO GET $x^2 \left(x^{-1/2} v'' - x^{-3/2} v' + \frac{3}{4} x^{-5/2} v \right) + x \left(x^{-1/2} v' - \frac{1}{2} x^{-3/2} v \right)$
$+ (x^2 - 1/4) x^{-1/2} v = 0$. THIS GIVES

$$x^{3/2} v'' + v' \left[-x^{+1/2} + x^{1/2} \right] + v \left[\frac{3}{4} x^{-1/2} - \frac{1}{2} x^{-1/2} + x^{3/2} - \frac{1}{4} x^{-1/2} \right]$$

CANCELLING TERMS: $x^{3/2} v'' + x^{3/2} v = 0 \rightarrow v'' + v = 0$. THUS

$v = C_1 \cos x + C_2 \sin x \longrightarrow y = C_1 x^{-1/2} \cos x + C_2 x^{-1/2} \sin x$.

PROBLEM 3 WE HAVE

$$x^2 y'' + 2xy' + (x^2 - 5/16)y = 0$$

NEAR $x = 0$ WE HAVE $x^2 y'' + 2xy' - 5/16\, y \cong 0$. PUT $y = x^r$ SO THAT

$$r(r-1) + 2r - 5/16 = 0 \rightarrow r^2 + r - 5/16 = 0 \rightarrow (r + 5/4)(r - 1/4) = 0$$

SO $r_1 = -5/4$, $r_2 = 1/4$. THUS WE WOULD LOOK FOR

$$y_1 = x^{1/4}[1 + a_1 x + a_2 x^2 + \dots], \quad y_2 = x^{-5/4}[1 + b_1 x + b_2 x^2 + \dots]$$

AND $y = c_1 y_1 + c_2 y_2 = c_1 x^{1/4}[1 + a_1 x + \dots] + c_2 x^{-5/4}[1 + b_1 x + b_2 x^2 + \dots]$.

NOW WE WANT $\lim_{x \to 0} y = 0$, $\lim_{x \to 0^+} x^{3/4} y' = 1$.

THE CONDITION $\lim_{x \to 0} y = 0 \rightarrow c_2 = 0$.

NOW $\lim_{x \to 0^+} x^{3/4} y' = \lim_{x \to 0^+} x^{3/4}\left[\frac{1}{4} c_1 x^{-3/4} + \dots\right] = 1$. THUS $c_1 = 4$.

AS SUCH WE NEED ONLY FIND A SOLUTION OF THE FORM

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \quad \text{WITH} \quad r = r_2 = 1/4 \quad \text{AND} \quad a_0 = 4.$$

THIS IS THE SOLUTION WE WANT. WE SUBSTITUTE TO GET

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} 2 a_n (n+r) x^{n+r} - \frac{5}{16} \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0.$$

WE GET $$\sum_{n=0}^{\infty} x^{n+r}\left[a_n (n+r)(n+r-1) + 2 a_n (n+r) - \frac{5}{16} a_n\right] + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0.$$

OR $$\sum_{n=0}^{\infty} x^{n+r}\left[(n+r)(n+r+1) - \frac{5}{16}\right] a_n + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0.$$

THIS GIVES

$$0 = a_0\left[r(r+1) - \frac{5}{16}\right]x^r + a_1\left[(1+r)(2+r) - \frac{5}{16}\right]x^{r+1} + \sum_{n=2}^{\infty} x^{n+r}\left[\left((n+r)(n+r+1) - \frac{5}{16}\right)a_n + a_{n-2}\right]$$

THUS $\quad r^2 + r - 5/16 = 0 \quad \rightarrow \quad r_1 = -5/4, \; r_2 = 1/4$

WE SET $\quad r = r_2 = 1/4 \quad$ AND THEN $\quad a_1\left[(1+r_2)(2+r_2) - 5/16\right] = 0 \rightarrow a_1 = 0.$

THE RECURSION IS $\quad a_n = \dfrac{-a_{n-2}}{(n+1/4)(n+5/4) - 5/16}, \quad n \geq 2.$

SO $\quad a_n = -\dfrac{a_{n-2}}{n(n+3/2)}, \quad n \geq 2.$

WE HAVE $\quad a_1 = a_3 = a_5 = \ldots = 0.$

NOW $\quad a_2 = -\dfrac{a_0}{2(7/2)} = -a_0/7$

$$a_4 = \frac{-a_2}{4[11/2]} = -\frac{a_2}{22} = \frac{a_0}{7 \cdot 22}$$

WE MUST TAKE $a_0 = 4$ AS EXPLAINED ON PREVIOUS PAGE. SO

$$\left\{ y = x^{1/4} a_0\left[1 - \frac{1}{7}x^2 + \frac{1}{7 \cdot 22}x^4 + \ldots\right] \right. \quad \text{WITH } a_0 = 4.$$

OR $\quad y = 4x^{1/4} - \dfrac{4}{7}x^{9/4} + \dfrac{2}{7 \cdot 11}x^{17/4} + \ldots$

PROBLEM 4 $\quad y'' + \frac{1}{\sin x} y' + \frac{(1-x)}{x^2} y = 0$

MULTIPLY BY x^2: $\quad x^2 y'' + x\left[\frac{x}{\sin x}\right] y' + (1-x) y = 0.$

THUS $\quad x^2 y'' + x\, p(x)\, y' + q(x) y = 0 \quad$ WITH $\quad p = \frac{x}{\sin x}, \quad q = 1-x.$

NOW AS $x \to 0$ WE SEE $\quad \lim_{x\to 0} p(x) = \lim_{x \to 0} \left(\frac{x}{\sin x}\right) = 1, \quad \lim_{x\to 0} q(x) = 1.$

THUS $\quad x^2 y'' + x y' + y \cong 0. \quad$ PUT $\quad y = x^r \quad$ SO $\quad r(r-1) + r + 1 = 0$

WHICH GIVES $\quad r^2 + 1 = 0 \quad$ SO $\quad r = \pm i. \quad$ THUS $\quad y = x^i = \cos(\ln x) + i \sin(\ln x)$

THEREFORE, WE DERIVE A SOLUTION OF THE FORM

$$y = x^i \sum_{n=0}^{\infty} c_n x^n \quad \text{WHERE } c_n \text{ ARE COMPLEX CONSTANTS}$$

TO BE FOUND. I.e. WE HAVE $\quad c_n = a_n + i b_n$. NOW DEFINE

$$y_1 = \mathrm{RE}[y] = \mathrm{RE}\left[x^i \sum_{n=0}^{\infty} c_n x^n\right] = \mathrm{RE}\left[\left(\cos(\ln x) + i \sin(\ln x)\right) \sum_{n=0}^{\infty} (a_n + i b_n) x^n\right]$$

$$y_2 = \mathrm{IM}[y] = \mathrm{IM}\left[\left(\cos(\ln x) + i \sin(\ln x)\right) \sum_{n=0}^{\infty} (a_n + i b_n) x^n\right].$$

THIS YIELDS

$$y_1 = \cos(\ln x) \sum_{n=0}^{\infty} a_n x^n - \sin(\ln x) \sum_{n=0}^{\infty} b_n x^n$$

$$y_2 = \cos(\ln x) \sum_{n=0}^{\infty} b_n x^n + \sin(\ln x) \sum_{n=0}^{\infty} a_n x^n$$

AND GENERAL SOLUTION IS $\quad y = d_1 y_1 + d_2 y_2$. <u>EQUIVALENTLY</u>, WE CAN LOOK FOR 2 independent solutions of the FORM

$$\hat{y}_1 = \cos(\ln x) \sum_{n=0}^{\infty} \hat{a}_n x^n \quad \text{AND} \quad \hat{y}_2 = \sin(\ln x) \sum_{n=0}^{\infty} \hat{b}_n x^n, \quad \hat{a}_n, \hat{b}_n \text{ REAL}$$

THEN GENERAL SOLUTION IS $\quad y = e_1 \hat{y}_1 + e_2 \hat{y}_2.$

PROBLEM 5

$$x^2 y'' + x y' + x^2 y = 0.$$

(i) WE KNOW ONE SOLUTION IS $y_1 = J_0(x)$. WE LOOK FOR THE SECOND SOLUTION AS $y = J_0(x) \ln x + \tilde{y}$.

WE SUBSTITUTE AFTER CALCULATING $y' = \frac{1}{x} J_0 + J_0' \ln x + \tilde{y}'$,

AND $y'' = -x^{-2} J_0 + 2x^{-1} J_0' + J_0'' \ln x + \tilde{y}''$.

THIS GIVES
$$x^2 [-x^{-2} J_0 + 2x^{-1} J_0' + J_0'' \ln x + \tilde{y}''] + x [\tfrac{1}{x} J_0 + J_0' \ln x + \tilde{y}'] + x^2 [J_0 \ln x + \tilde{y}] = 0.$$

GROUPING TERMS:
$$x^2 \tilde{y}'' + x \tilde{y}' + x^2 \tilde{y} + (\ln x)[J_0'' x^2 + x J_0' + x^2 J_0] + J_0[-1+1] + 2x J_0' = 0.$$

NOW $x^2 J_0'' + x J_0' + x^2 J_0 = 0$ GIVES THAT

$$(*) \qquad x^2 \tilde{y}'' + x \tilde{y}' + x^2 \tilde{y} = -2x J_0'(x)$$

(ii) NOW WE PUT $J_0(x)$ INTO THE DIFFERENTIAL EQUATION:

$$J_0(x) = 1 - \frac{1}{2^2} x^2 + \frac{1}{2^2 \cdot 4^2} x^4 - \frac{1}{2^2 4^2 6^2} x^6 + \dots$$

NOW PUT $J_0' = -\frac{1}{2} x + \frac{1}{2^2 \cdot 4} x^3 - \frac{1}{2^2 \cdot 4^2 \cdot 6} x^5$

THUS
$$x^2 \tilde{y}'' + x \tilde{y}' + x^2 \tilde{y} = x^2 - \frac{1}{8} x^4 + \frac{x^6}{2^6 \cdot 3} + \dots$$

NOW WE PUT IN A SERIES
$$\tilde{y} = \sum_{n=0}^{\infty} b_n x^n.$$

SINCE RHS IS EVEN IN x, WE MUST HAVE THAT $\tilde{y}$ IS EVEN IN x.

WE PUT $\tilde{y} = b_0 + b_2 x^2 + b_4 x^4 + b_6 x^6 + \dots$

WE SUBSTITUTE:

$$x^2\left[2b_2 + 12\,b_4\,x^2 + 30\,b_6\,x^4 + \dots\right] + x\left[2b_2\,x + 4\,b_4\,x^3 + 6\,b_6\,x^5 + \dots\right]$$
$$+\ x^2 b_0 + x^4 b_2 + b_4 x^6 + \dots = x^2 - \frac{1}{8}x^4 + \frac{x^6}{2^6 \cdot 3}$$

EQUATING x^2: $\quad 2b_2 + 2b_2 + b_0 = 1$

x^4: $\quad 12\,b_4 + 4b_4 + b_2 = -\frac{1}{8}$

x^6: $\quad 30\,b_6 + 6b_6 + b_4 = \frac{1}{2^6 \cdot 3}$

NOW WLOG WE TAKE $b_0 = 0$. THEN $b_2 = 1/4$.

$$16\,b_4 = -\frac{1}{8} - \frac{1}{4} = -\frac{3}{8}.$$

THUS $\quad b_4 = -\frac{3}{8 \cdot 16} = -\frac{3}{2^7}$

THEN $\quad 36\,b_6 = \frac{1}{2^6 \cdot 3} + \frac{3}{2^7} = \frac{2}{2^7 \cdot 3} + \frac{9}{2^7 \cdot 3} = \frac{11}{2^7 \cdot 3}.$

THUS $\quad b_6 = \frac{11}{2^7 \cdot 3 \cdot 36} = \frac{11}{2^9 \cdot 3^3}.$

THUS $\quad \hat{y} = b_2\,x^2 + b_4\,x^4 + b_6\,x^6 = x^2/4 - 3/27\,x^4 + \frac{11}{2^9 \cdot 3^3}x^6 + \dots$

THIS GIVES,

$$y = J_0(x)\ln x + \tilde{y} = (\ln x)\left[1 - x^2/4 + x^4/2^2 \cdot 4^2 - 1/2^2 \cdot 4^2 \cdot 6^2\,x^6 + \dots\right]$$
$$+\ x^2/4 - 3x^4/27 + 11/2^9 \cdot 3^3\,x^6 + \dots$$