

Topics for Test 2

filename: `test2topics.tex`

November 13, 2010

Branch points and branch cuts

(1.1) Be able to give the definition of a branch point. (1.2) Given a multi-valued function, be able to decide whether a given point in the complex plane or the point at infinity is a branch point. (1.3) Be able to explain what a branch cut is and how branch cuts are used to define single-valued branches of a multi-valued function.

(2.1) Be able to define Log , the principal branch of the logarithm and be able to compute the values $\text{Log}(z)$ for any $z \in \mathbb{C}$. (2.2) Be able to define branches of the logarithm with branch cuts along any ray joining 0 to ∞ , and be able to compute the values of the resulting functions.

(3.1) Be able to compute branches of $p(z)^\alpha$ where $p(z)$ is a polynomial whose factorization $p(z) = (z - r_1)(z - r_2) \cdots (z - r_n)$ is known or can be computed. (3.2) Be able to determine the regions of analyticity for the branches defined, and in examples be able to define branches that are analytic inside or outside given regions in the plane.

(4.1) Be able to define branches of inverse trig functions, including the principal branch, and determine the regions of analyticity.

Integration

(1.1) Be able to write down the defintion of the complex integral $\int_C f(z)dz$ where C is a contour parametrized by $z(t)$, $t \in [a, b]$ or a union of contours $C = C_1 \cup C_2 \cup \dots \cup C_n$. Be able to evaluate the integral from the definition in simple cases. (1.2) Be able to find parametrizations for contours such as straight line segments, circular arcs and parabolas. (1.3) Be able to use simple properties such as linearity, sign change under reversal of contour orientation, the basic bound $|\int_C f(z)dz| \leq (\max_{z \in c} |f(z)|) \times \text{length}(C)$ to evaluate and estimate complex integrals.

(2.1) Be able to evaluate integrals $\int_C f(z)dz$ using the fundamental theorem of calculus if the antiderivative $F(z)$ is known in a region including C . (2.2) In a case where the antiderivative is a multivalued function, be able to choose a branch that is suited to the contour. (2.3) For a region D in the complex plane and $f(z)$ defined in D , be able to demonstrate the equivalence of (i) the existence of an antiderivative $F(z)$ in D , (ii) the value of zero for the integral of $f(z)$ around any closed loop in D , and (iii) the path independence of complex integrals of $f(z)$ over paths in D .

Cauchy's integral theorem, Cauchy's integral formula and consequences

(1.1) Be able to state Cauchy's integral theorem and show how it follows from Green's theorem in the plane (under the additional hypothesis that $f'(z)$ is continuous). (1.2) Be able to derive consequences of Cauchy's theorem such as (i) analytic functions in simply connected regions have antiderivatives, (ii) invariance of integrals under deformation of contours.

(2.1) Be able to state Cauchy's integral formula and show how it follows from the Cauchy integral theorem. (2.2) Use Cauchy's integral formula to evaluate integrals around closed loops of quotients $f(z)/p(z)$ where $f(z)$ is analytic and $p(z)$ is a polynomial (and integrals that can be transformed to an integral of this form). (2.3) Be able to state the generalized Cauchy integral formula for derivatives $f^{(n)}(z)$. (2.4) Be able to state Liouville theorem and show it follows from the Cauchy integral formula for $f'(z)$ together with estimates on the integral appearing in this formula. (2.5) Be able to show how the fundamental theorem of algebra follows from Liouville's theorem.

(3.1) Be able to state the mean value property for analytic functions. (3.2) Be able to state the maximum modulus principle for analytic functions as well as harmonic functions and show how it follows from the mean value property. (3.3) Be able to use the maximum modulus principle to find the maximum value of the modulus of an analytic (or harmonic) function on a domain by restricting attention to the boundary of the domain. (3.4) Be able to use the maximum modulus principle, together with estimates on the boundary of a domain, to show that certain analytic functions have no zeros inside the domain.