# Representations of complex Lie algebras and Weyl character formula. Part 1.

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October 1st, 2019

## 1. Representation of semisimple Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra.

Recall : by Lie's Theorem : If $\mathfrak{g}$ is solvable, and $V$ is a representation of $\mathfrak{g}$ then there is $v \in V$ nonzero such that $v$ is an eigenvector of $X$ for all $X \in \mathfrak{g}$.

A consequence : Let $\mathfrak{g}_{ss} = \mathfrak{g}/\mathrm{Rad}\mathfrak{g}$ where $\mathrm{Rad}(\mathfrak{g})$ is the maximal sovlable subalgebra. Every irrep of $\mathfrak{g}$ is of the form $V_0 \otimes L$ where $V_0$ is an irreducible representation of $\mathfrak{g}_{ss}$ and $L$ is 1-dimensional.

Upshot : we can often pass representations to the semisimple part.

Question : How do we study representations of $\mathfrak{g}$, fix $\mathfrak{g}$ to be semisimple.

**Step 1 :** Find a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, i.e. a maximal abelian maximal diagonalizable subalgebra of $\mathfrak{g}$.

**Step 2 :** $\mathfrak{h}$ acts on $\mathfrak{g}$ via adjoint representation, we get the Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha \right).$$

Same for representations $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$ (weight space decomposition).

We define

- *roots* of $\mathfrak{g}$ : $\alpha \in \mathfrak{h}^* \setminus \{0\}$ st $\mathfrak{g}_\alpha \neq 0$.
- *root space* : $\mathfrak{g}_\alpha$
- $R = \{\alpha \in \mathfrak{h}^* \text{ roots}\}$
- $R$ generate a lattice $\Lambda_R \subset \mathfrak{h}^*$ of rank $\dim \mathfrak{h}$. Call this *root lattice.*
- If $V$ is a representation then call $\dim V_\alpha$ the *multiplicity* of $\alpha$
- $\mathfrak{I}_\beta : V_\alpha \to V_{\alpha+\beta}$

All weights of irreps are congruent modulo $\Lambda_R$.

**Step 3 :** Find distinguished subalgebra $s_\alpha \simeq \mathfrak{sl}_2\mathbb{C} \subset \mathfrak{g}$ for each $\alpha$.

set

$$s_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \cong \mathfrak{sl}_2.$$

Pick a basis $X_\alpha, Y_\alpha, H_\alpha = [X_\alpha, Y_\alpha]$.

$H_\alpha$ is determined by $X_\alpha, Y_\alpha$ and the requirement that $\alpha(H) = 2$

**Step 4 :** A consequence based on representations of $\mathfrak{sl}_2\mathbb{C}$, any representation of $\mathfrak{g}$ have *integral* eigenvalues at each $H_\alpha$

Define the *weight lattice*

$$\Lambda_\omega = \{\beta \in \mathfrak{h}^* : \beta(H_\alpha) \in \mathbb{Z}\ \forall \alpha \in R\}.$$

**Step 5 :** Account for the symmetry of $\mathfrak{sl}_2\mathbb{C}$ representations.

For $\alpha \in R$ and $\beta \in \mathfrak{h}^*$ we define

$$\mathcal{W}_\alpha(\beta) = \beta - \frac{2\beta(H_\alpha)}{\alpha(H_\alpha)}\alpha = \beta = \beta(H_\alpha)\alpha.$$

$\mathcal{W}_\alpha$ reflects the lines spanned by $\alpha$ through the hyperplane $SL_\alpha = \{\langle H_\alpha, \beta\rangle = 0\}$

The *Weyl group* of $\mathfrak{g}$ is

$$\mathcal{W} = \langle \mathcal{W}_\alpha : \alpha \in R\rangle$$

**Fact.** Set of weights of any representation and its multiplicities is invariant under $\mathcal{W}$.

**Step 6 :** Choose a “direction” in $\mathfrak{h}^*$ i.e. choose a real linear funcitonal $\ell$ on $\Lambda_R$ giving a decomposition $R = \underbrace{R^+}_{\text{positive roots}} \cup \underbrace{R^-}_{\text{negative roots}}$.

Define a *highest weight vector* of $V$ (rep of $\mathfrak{g}$) to be an eigenvector $v$ such that $v$ is in the kernel of $\mathfrak{g}_\alpha$ for all $\alpha \in R^+$. The highest weight of $v$ to be othe *highest weight*.

**Fact.**

- Every finite-dimensional representation has a highest weight vector.
- Every finite-dimensional irreducible representation has a unique highest weight up to scalar multiple.
- Subspace $W$ of $V$ generated by application $\mathfrak{g}_\beta$, $\beta \in R^-$ on a highest weight vector is irreducible

- Every vertex of the convex hull of weights of $V$ is conjugate to a highest weight $\alpha$ under $\mathcal{W}$.
- $\alpha(H_\gamma) \geq 0$ for all $\gamma \in R^+$. The locus of these inequalities is a (closed) WEyl chamber.

We get the set of weights of $V$ as erights congruent to a highest weight $\alpha$ modulo $\Lambda_R$ and lie in the convex hull of images of $\alpha$ under $\mathcal{W}$.

**Theorem.** For any $\alpha$ in the intersection between the Weyl chamber with $\Lambda_\omega$, there exists a unique finite dimensional irrep $\Gamma_\alpha$ of $\mathfrak{g}$ with highest weight $\alpha$.

We get a bijection

$$\text{Weyl chamber} \cap \Lambda_{\mathcal{W}} \longleftrightarrow \text{f.d. irrep of } \mathfrak{g}$$
$$\alpha \longmapsto \Gamma_\alpha.$$

Question : how to get multiplicities ?

For $\mathfrak{sl}_n$ :

Step 1 : $\mathfrak{h} = \{\sum_i a_i H_i, \sum a_i = 0\}$, $\mathfrak{h}^* = \mathbb{C}[L_1, \cdots, L_n]/\langle \sum L_i \rangle$

$L_i(H_i) = 1$.

Step 2 : $F_{ij}$ are eigenvectors with root $L_i - L_j$. Roots $R = \{L_i - L_j; i \neq j\}$.

root lattice : $\lambda_R = \{\sigma a_i L_i, \sum a_i = 0, a_i \in \mathbb{Z}\}$

Step 3 : $s_{L_i - L_j}$ is generated by $E_{ij}, E_{ji}$, and $H_i - H_j$.

Step 4 : $\sum a_i L_i \in \Lambda_\omega \Leftrightarrow a_\ell \equiv a_k \bmod \mathbb{Z}$ for all $k, \ell$.

Step 5 : $\mathcal{W}_{L_i - L_j}$ switches $L_i$ and $L_j$ and fixes everything else. $\mathcal{W} = S_n$.

$R^+ = \{L_i - L_j : i < j\}$

$R^- = \{L_i - L_j : i < j\}$

Weyl chamber $= \{\sum a_i L_i : a_1 \geq \cdots \geq a_n\}$

For $\mathfrak{sl}_3$ :

irrep $\Gamma_{a,b}$ weight $aL_1 - bL_3$, $a, b \in \mathbb{N}$.

$V = \mathbb{C}^3$ : eigenvalues $\{L_1, L_2, L_3\}$

$V^* : $ eigenvalues are $\{-L_1, -L_2, -L_3\}$.

$\mathrm{Sym}^2(V)$, the eigenvalues are $\{2L_i, L_i + L_j\}$.

$\mathrm{Sym}^2(V) \otimes V^*$ eigenvalues $= \{2L_i - L_j, L_i + L_j - L_k, L_i\}$

$$\mathrm{Sym}^2(V) \otimes V^* \to \mathrm{Sym}^2(V)$$

$$vw \otimes u^* \mapsto \langle v, u^* \rangle w + \langle w, u^* \rangle v.$$

Kernel of this map is $\Gamma_{2,1}$ so $\mathrm{Sym}^2 \otimes V^* = \Gamma_{2,1} \oplus V$.

Back to $\mathfrak{sl}_n\mathbb{C}$.

$V = \mathbb{C}^n$

$V$ has highest weight $L_1$

$\mathrm{Sym}^m V$ has highest weight $mL_1$

$\bigwedge^m V$ has highest weight $L_1 + \cdots + L_m$.

irreps of $\mathfrak{sl}_n\mathbb{C}$ are $\Gamma_{a_1,\ldots,a_{n-a}}$ with highest weight $(a_1 + \cdots + a_{n-1})L_1 \cdots + a_{n-1}L_{n-1}$.

$\Gamma_{a_1,\ldots,a_{n-1}} \subset \mathrm{Sym}^{a_1} V \otimes \mathrm{Sym}^{a_2} \bigwedge^2 V \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}} \bigwedge^{n-1} V$.

Question : How do you describe $\Gamma_{a_1,\ldots,a_{n-1}}$ ?

**Weyl construction**

Let $V$ be an $n$-dimensional $\mathbb{C}$-vector space.Consider the natural action of $S_d$ on $V^{\otimes d}$.

**Def.** Let $\lambda = \lambda_1 \geq \cdots \geq \lambda_n$ be a partition of $d$

A *Weyl module* or *Weyl contruction* associated to $\lambda$ of a $\mathbb{C}$-vector space is

$$\mathbb{S}_\lambda V := V^{\otimes d} \otimes_{\mathbb{C}S_d} \tilde{V}_\lambda,$$

where $\tilde{V}_\lambda$ is the irrep of $S_d$ associated to $\lambda$ (section 6 of Fulton-Harris).

Fact : Any endomorphism $g$ of $V$ lidts to an endomorphism of $\mathbb{S}_\lambda V$. Look at the character of $\mathbb{S}_\lambda V$

$\chi_{\mathbb{S}_\lambda V}$ = trace of (image of) $g$

**Theorem.** $\chi_{\mathbb{S}_\lambda V}(g) = S_\lambda(x_1, \cdots, x_n)$ where $x_i$ are eigenvalues of $g$

$$\dim \mathbb{S}_\lambda V = S_\lambda(1, \cdots, 1) = \prod_{i,j} \frac{\lambda_i - \lambda_j + (j-i)}{j-i}$$

Those $S_\lambda$ are called *Schur polynomials.*

Brief detour to symmetric polynomials

Write $M_\mu(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} x_\sigma(1)^{\mu_1} \cdots x_{\sigma(n)}^{\mu_n}$.

For example $M_{311}(x_1, x_2, x_3) = 2x_1^3 x_2 x_3 + 2x_! x_2^3 x_3 + 2x_1 x_2 x_3^3$.

$H_d = \sum_{1 \leq i_1 \leq \cdots \leq i_d \leq n} x_{i_1} \cdots x_{i_d}$, $E_d = \sum_{1 \leq j_1 < \cdots < j_d \leq k} x_{j_1} \cdots x_{j_d}$.

*Schur polynomials* are a specific basis for the algebra of symmetric functions,

$S_\lambda(x_1, \cdots, x_n) = \sum_{T \in SSYT(\lambda)} x^T$ where $x^T = x^{T_1} \cdots x^{T_n}$ and $SSYT(\lambda)$ is the set of semistandard Young Tableau of shape $\lambda$ (semistandard = can have repeated numbers).

**Facts.** + $s_\lambda = \sum_{\mu \leq \lambda} K_{\lambda\mu} m_\mu$.

$K_{\lambda\mu}$ are Kostka numbers : number of SSYT of shape $\lambda$ with weightt $\mu$.

- Jacobi bialternant formula :

$$[X^{\mu_i}] = \begin{bmatrix} X_1^{\mu_1} & \cdots & X_n^{\mu_1} \\ & \vdots & \\ X_1^{\mu_n} & \cdots & X_n^{\mu_n} \end{bmatrix}$$

Then $S_\lambda = \frac{\det[X^{\lambda_i - n - i}]}{\det[X^{u-i}]}$. Note that the denominator is a Vandermonde determinant.

Back to $\mathfrak{sl}_n$.

Try to apply Weyl's construction on $V = \mathbb{C}^n$. $\mathbb{S}_\lambda V$ can be seen as a rep of $\mathrm{SL}_n(\mathbb{C})$ and get a derived action of $\mathfrak{sl}_n$.

**Prop.** $\mathbb{S}_\lambda \mathbb{C}^n$ is the irrep of $\mathfrak{sl}_n\mathbb{C}$ if highest weight $\lambda_1 L_1 + \cdots + \lambda_n L_n$.

"Proof." $S_\lambda = \sum_{\mu \leq \lambda} K_{\lambda\mu} m_\mu$ where $K_{\lambda\lambda} = 1$ and $m_\mu$ corresponds to weight $\sum \lambda_i L_i$.

**Rem :** $\mathbb{S}_\lambda V \approx \mathbb{S}_\mu V$ if and only if $\lambda_i - \mu_i = C$.

So $\Lambda_{a_1, \ldots, a_{n-1}} \to \mathbb{S}_{a_1 + \cdots + a_{n-1}}, \ldots, a_{n-1}$

**Cor.** $\dim \Gamma_{a_1, \cdots, a_{n-1}} = \prod_{1 \leq i < j \leq n} \frac{a_i + \cdots a_{j-1} + (j-i)}{j-i}$.

**Facts.**

- $\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu V = \oplus_\nu c_{\lambda\mu}^\nu \mathbb{S}_\nu \mathbb{S}_\nu(V)$ where $C_{\lambda\mu}^v$ is the Littlewood-Richardson coefficient.