Geometric Satake equivalence

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2. Geometric Satake equivalence

2.1. Affine Grassmannians

$\mathcal{O} = \mathbb{C}[[\varpi]]$

$\mathcal{K} = \mathbb{C}((\varpi))$

H complex algebraic group

$H_{\mathcal{O}}$ = $\mathbb{C}$-group scheme which represents the functor $R \mapsto H(\mathbb{C}[[\varpi]])$ ($= L^+H$ in Timo's lecture). $H_{\mathcal{K}}$ = $\mathbb{C}$-group scheme which represents the functor $R \mapsto H(\mathbb{C}((\varpi)))$ ($= LH$ in Timo's lecture).

From now on G is a complex connected reductive algebraic group

B is a Borel subgroup

T maximal torus

B^- is the opposite Borel subgroup

N = unipotent radical of B

$N^- = $ unipotent radical of B^-

W Weyl group of (G, T)

$\mathbb{X}^\vee = \mathbf{X}_\star(T)$ cocharacter of T

simple coroots $= \Delta_s^\vee = \Delta_s^\vee(G, B, T) \subset$ positive roots $= \Delta_+^\vee = \Delta_+^\vee(G, B, T) \subset \Delta^\vee = \Delta^\vee(G, T)$ coroots of (G, T)

$\mathbb{X}_+^\vee$: dominant characters

Same for $\mathbb{X} \supset \Delta \supset \Delta_+ \supset \Delta_s$

Dominance order on $\mathbb{X}^\vee$. $\lambda, \mu \in \mathbb{X}^\vee$.

$\lambda \leq \mu \Leftrightarrow \lambda - \mu \in \mathbb{Z}_{\geq 0}\Delta_+^\vee$.

ρ = halfsum of positive roots ($\langle \rho, - \rangle : \mathbb{X}^\vee \to \frac{1}{2}\mathbb{Z}$)

Affine Grassmannian $\mathrm{Gr}_G = ((G_{\mathcal{K}}/G_{\mathcal{O}})_{et})_{red}$ ind-reduced, ind-proper ind-scheme, ind-(of finite type).

2.2 Decompositions

The embedding $T \subset G$ induces a closed embedding $\mathrm{Gr}_T \to \mathrm{Gr}_G : \varpi^\lambda T_{\mathcal{O}} \mapsto L_\lambda$.

$\mathrm{Gr}_T = \mathbb{X}^\vee$ via $\lambda \mapsto \varpi^\lambda T_{\mathcal{O}}$.

Cartan decomposition. $\mathrm{Gr}_G = \sqcup_{\lambda \in \mathbb{X}^\vee_+} \mathrm{Gr}^\lambda_G$ with $\mathrm{Gr}^\lambda_G = \mathcal{O} \cdot L_\lambda$. (smooth locally closed subvariety).

We have $\overline{\mathrm{Gr}^\lambda_G} = \bigsqcup_{\lambda \in \mathbb{X}^\vee_+ \ \mu \leq \lambda} \mathrm{Gr}^\mu_G$ (proj var with algebraic stratification)

$\dim(\mathrm{Gr}^\lambda_G) = \langle 3p, \lambda \rangle$

P^-_λ = parabolic subgroup of G containing B^- and associated with $\{\alpha \in \Delta_s | \langle \lambda, \alpha \rangle = 0\}$.

Then we have $\mathrm{Gr}^\lambda_G \to G/P^-_\lambda$ via $L_\lambda \mapsto P^-_\lambda$. For $\lambda \in \mathbb{X}^\vee_+$ This is a Zariski locally trivial fibration whose fibers are affine spaces.

Consequences. Gr^λ_G is simply connected (no nontrivial local systems)

Bruhat decomposition . $I \subset G_0$ Iwahori subgroup $\to B \subset G$ via $\varpi \mapsto 0$.

Then $\mathrm{Gr}_G = \bigsqcup_{\lambda \in \mathbb{X}^\vee} \mathrm{Gr}_{G,\lambda}$ with $\mathrm{Gr}_{G,\lambda} = I \cdot L)\lambda$ (isom. to an affine space).

For $\lambda \in \mathbb{X}^\vee_+$ we have

$$\mathrm{Gr}^\lambda_G = \bigsqcup_{\mu \in W \cdot \lambda} \mathrm{Gr}_{G,\mu}$$

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$$G/P^-_\lambda = \bigsqcup_{w \in W/W_\lambda} BwP^-_\lambda/P^-_\lambda \ (\mu = w\lambda).$$

Iwasawa Decomposition.

$\mathrm{Gr}_G = \bigsqcup_{\lambda \in \mathbb{X}^\vee} S_\lambda$ with $S_\lambda = N_{\mathcal{K}} \cdot L_\lambda$

$= \bigsqcup_{\lambda \in \mathbb{X}^\vee} T_\lambda$ with $T_\lambda = N^-_{\mathcal{K}} \cdot L_\lambda$.

Both S_λ and T_λ are ind-varieties.

$\overline{S_\lambda} = \bigsqcup_{\nu \in \mathbb{X}^\vee \ \nu \leq \lambda} S_\nu$

$\overline{T_\lambda} = \bigsqcup_{\nu \in \mathbb{X}^\vee \ \nu \geq \lambda} T_\nu$

2.3. The Satake Category

k commutative Noetherian ring of finite global dimension.

Satake Category. $\mathbf{Perv}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, k)$ $G_{\mathcal{O}}$-equivariant. k perverse sheaves on Gr_G with respect to the stratification by G_0-orbits.

Pf here. Gr_G is an ind-variety and not a variety. G_0 is not of finite type.

One overcomes these difficulties in the following way :

if $X \subset \mathrm{Gr}_G$ is a finite union of G_0 orbits, then X is a (proj) variety. Moreover, the G_0-action on X factors through the action of $L_i^+ G$ for $i >> 0$.

Fact. The category $\mathbf{Perv}_{L_i^+ G}(X, k)$ does **not** depend on the choice of i.

Then we set $\mathbf{Perv}_{G_0}(\mathrm{Gr}_G, k) = \varinjlim_{X} \mathbf{Perv}_{G_{\mathcal{O}}}(X, k)$ where X runs over finite closed unions of $G_{\mathcal{O}}$-orbits.

Remark. If $X_q \subset X_2$, $\mathbf{Perv}_{G_{\mathcal{O}}}(X_1, k) \to \mathbf{Perv}_{G_{\mathcal{O}}}(X_2, k)$ is fully faithful so there are no subtleties in the colimit. Below we will ignore those subtleties.

2.4 Convolution

We consider the twisted product

$$\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G = ((G_{\mathcal{K}} \times \mathrm{Gr}_G)/G_{\mathcal{O}})_{et,red}$$

we have $m : \mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G \to \mathrm{Gr}_G$ induced by $(g, hG_{\mathcal{O}}) \mapsto ghG_{\mathcal{O}}$.

Prop (Mirkovic - Vilonen). m is stratified semismall with respect to the stratifications $(\mathrm{Gr}_G^\lambda \tilde{\times} \mathrm{Gr}_G^\mu)_{\lambda, \mu \in \mathbb{X}_+^\vee}$ and $(\mathrm{Gr}_G^\lambda)_{\lambda \in \mathbb{X}_+^\vee}$.

FOr $\mathcal{F}, \mathcal{G} \in \mathbf{Perv}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, k)$ we consider $p^*(\mathcal{F})^L \boxtimes_k \mathcal{G}) \in \mathbf{Perv}(G_{\mathcal{K}} \times \mathrm{Gr}_G, k)$.

$p : G_{\mathcal{K}} \to \mathrm{Gr}_G$ projection. This is a $G_{\mathcal{O}}$ equivariant perverse sheaf (for the diagonal $G_{\mathcal{O}}$ action). So by descent there exists a perverse sheaf $\mathcal{F} \tilde{\boxtimes} \mathcal{G}$ on $\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G$ whose pullback to $G_{\mathcal{K}} \times \mathrm{Gr}_G$ is $p^* \mathcal{F}^L \boxtimes_k \mathcal{G}$, take

$$\mathcal{F} \star \mathcal{G} := m_*(\mathcal{F} \tilde{\boxtimes} \mathcal{G}) \; textperversesheafbystratifiedsemismallness.$$

Facts.

- Convolution is *associative* (i.e. there eists a canonical isom $(- \star -) \star - = - \star (- \star -)$ functorial in each entry).
- The object $\delta_{\mathrm{Gr}} :=$ skyscraper sheaf at $L_0 \in \mathrm{Gr}_G$ is a unit object (i.e. there are canonical isom $\delta_{\mathrm{Gr}} \times - \;\simeq\;$ id, $d \simeq \; - \star \delta_{\mathrm{Gr}}$)

So it is a *monoidal* category.

2.5. Statement

$G_k^\vee$ = "Langlands dual reductive k-group" = $\mathrm{Spec}(k) \times_{\mathrm{Spec}(\mathbb{Z})} G_\mathbb{Z}^\vee$ where $G_\mathbb{Z}^\vee$ is the unique split reductive group over $\mathbb{Z}$ whose base change to $\mathbb{C}$ whose root datum is $(\mathbb{X}^\vee, \mathbb{X}, \Delta^\vee, \Delta)$ (exchange roots and coroots).

$\mathrm{Rep}(G_k^\vee)$= cat of algebraic $G_k^\vee$-modules ($\mathcal{O}(G_k^\vee)$-comodule) which are finitely generated as k-modules.

Theorem. There exists an equivalence of monoidal categories $(\mathbf{Perv}_{G_0}(\mathrm{Gr}_G, k), \star) \simeq (\mathrm{Rep}(G_k^\vee), \otimes)$ under which the forgetful functor $\mathrm{Rep}(G_k^\vee) \to \mathrm{Mod}_k^{fg}$ corresponds to $\mathbb{H}^*(\mathrm{Gr}_G, -) : \mathbf{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G, k) \to \mathbf{Mod}_k^{fg}$.

Remarks.

- (1.1) Simple objects (in case k is an algebraically closed field) in $\mathrm{Rep}(G_k^\vee)$ are classified by highest weights (in $\mathbb{X}_+^\vee$).
- (1.2) In $\mathbf{Perv}_{G_0}(\mathrm{Gr}, k)$: classified by pairs $(\mathrm{Gr}_G^\lambda, \mathcal{L})$. Here $\mathcal{L}$ must be $\underline{k}$. The simple objects are parametrized by $\mathbb{X}_+^\vee$.
- (2) Assume further that $\mathrm{char}(k) = 0$. Then we will see later that $\mathbf{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G, k)$ is semisimple. The same is true for $\mathrm{Rep}(G_k^\vee)$.

The existence of an equivalence $\mathrm{Rep}(G_k^\vee) \simeq \mathbf{Perv}_{G_\mathcal{O}}(\mathrm{Gr}_G, k)$ is obvious. The main content of the theorem is then the compatibility with monoidal structures.

- (3) We will do slightly better. We will construct a group scheme $\tilde{G}_k$ for any k and an equivalence $\mathbf{Perv}_{G_0}(\mathrm{Gr}_G, k) \simeq \mathrm{Rep}(\tilde{G}_k)$ such that $\tilde{G}_{k'} \simeq \mathrm{Spec}(k') \times_{\mathrm{Spec} k} \tilde{G}_k$ for any $k \to k'$ and show that $\tilde{G}_\mathbb{Z}$ is split reductive (with a canonical maximal torus) with appropriate root datum.

2.6. Commutativity

The tensor product in $\mathrm{Rep}(G_k^\vee)$ is commutative, i.e. for $M, N \in \mathrm{Rep}(G_k^\vee)$ we have a canonical isomorphism $M \otimes_k N \xrightarrow{\sim} N \otimes_k M$ so if the theorem is true, the same should hold for $\mathrm{Perv}_{G_\mathcal{O}}(\mathrm{Gr}, k)$.

In fac tthe proof will require to construct such an isomorphism before proving the theorem.

Idea of the construction. (Drinfeld) Use the moduli representation of Gr_G. We set $G = \mathbb{A}_\mathbb{C}^1$, $C^\times = \mathbb{Z}_\mathbb{C}^1 \backslash \{0\}$. Recall from Timo's lecture that Gr_G represents the functor $R \mapsto \{(\mathcal{E}, \beta) | \mathcal{E}\ G-\text{bundle on } C_R = C \times \mathrm{Spec} R\ \beta : \mathcal{E}^\circ_{C_R^\times} \xrightarrow{\sim} \mathcal{E}|_{C_R^\times}\}/\simeq$.

"global" version. $\mathrm{Gr}_{G,C} \to C$ ind-scheme which represents the functor

$$R \mapsto \left\{(y, \mathcal{E}, \beta) | y \in C(\mathbb{R})\ \mathcal{E}\ G\text{-bundle on } C_R\ \beta : \mathcal{E}^\circ_{C_R \backslash \Gamma_y} \xrightarrow{\sim} \mathcal{E}_{C_R \backslash \Gamma_y}\right\}.$$

But one can do that also over C^2 : one gets the Beilinson-Drinfeld Grassmannian Gr_{G,G^2} : ind-scheme over C^2 which represents

$$R \mapsto \left\{ (y_1, y_2, \mathcal{E}, C) | y_!, y_2 \in C(R) \ \mathcal{E} \ G\text{-bundle over } C_R \ \beta : \mathcal{E}^\circ_{C_R \backslash (\Gamma_{y_1} \cup \Gamma_{y_2})} \xrightarrow{\sim} \mathcal{E}_{C_R \backslash (\Gamma_{y_1} \cup \Gamma_{y_2})} \right\}.$$

Facts.

- (1) This functor is represented by and ind-proper ind-scheme over C^2
- (2) $\mathrm{Gr}_{G,C^2} \times_{C^2} \Delta C = \mathrm{Gr}_{G,C} \times_C \Delta C = \mathrm{Gr}_G \times \Delta C$
- (3) $\mathrm{Gr}_{G,C^2} \times_{C^2} (C^2 \backslash \Delta C) = (\mathrm{Gr}_{G,C} \times \mathrm{Gr}_{G,C})|_{C^2 \backslash \Delta C} \simeq \mathrm{Gr}_G \times \mathrm{Gr}_G \times (C^2 \backslash \Delta C)$.

To $\mathcal{E}$ one associates the pair $(\mathcal{E}_1, \mathcal{E}_2)$ where $\mathcal{E}_i$ is the G-bundle obtained by glueing $\mathcal{E}^\circ_{C_R \backslash \Gamma_{y_i}}$ using the trivialization β $(j \neq i)$.

We set $i : \mathrm{Gr}_G \times \Delta C \to \mathrm{Gr}_{G,C^2}$ (closed embedding)

$j : \mathrm{Gr}_G \times \mathrm{Gr}_G \times C^2 \backslash \Delta C \to \mathrm{Gr}_{G,C^2}$ (open embedding)

Theorem (Belinson - Drinfeld). There exists an isomorphism

$$i^* \left(j_{!*} {}^p\mathcal{H}^0 \left({\mathcal{F}_1}^L \boxtimes_k {\mathcal{F}_2}^L \boxtimes_k \underline{k}_{C^2 \backslash \Delta C}[2] \right) \right)^{[-1]} \simeq (\mathcal{F}_1 \times \mathcal{F}_2)^L \boxtimes_k \underline{k}_{\Delta C^2}[1].$$

The construction on the left handside is called the *fusion product.*

Application. We have $\sigma : \mathrm{Gr}_{G,C^2} \xrightarrow{\sim} \mathrm{Gr}_{G,C^2}$ obtained by switching y_1 and y_2. Restrict trivially to ΔC and to $(gG_\mathcal{O}, hG_\mathcal{O}, y_1, y_2) \mapsto (hG_\mathcal{O}, gG_\mathcal{O}, y_1, y_2)$ over $C^2 \backslash \Delta C$. Using the fact that $i^* \simeq i^* \sigma^*$ (because $\sigma_i = i$) one obtains a canonical isomorphism $\mathcal{F}_1 \star \mathcal{F}_2 \xrightarrow{\sim} \mathcal{F}_2 \star \mathcal{F}_1$.

Remark. On fact one needs to twist this isomorphism by a sign depending on the connected component supporting $\mathcal{F}_1$ and $\mathcal{F}_2$ to get the actual commutativity contraint.