Geometric Satake equivalence

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References.

- Mirkovic– Vilonen “*Geometric Langlands duality and representations of algebraic groups over commutative rings*”.
- Baumann–R. “*Notes on the geometric Satake equivalence*”

Plan.

- **Lecture 1 :** Constructible sheaves.
- **Lecture 2 :** Statement of the equivalence
- **Lecture 3 :** Proof (for general coefficients)
- **Lecture 4 :** End of the proof tilting modules and parity sheaves

1. Brief reminder on constructible sheaves.

Ref.

- Kashiwa?? –Schapira “*Sheaves on manifolds*”
- P. Achar *Lecture notes on perverse sheaves*"
- Chiss- Ginzburg “*Rep theory and complex geometry*”

1.1 Constructible derived category.

X complex algebraic variety.

Def. An *algebraic stratification* of X is a finite partition

$$X = \sqcup_{s\in\mathcal{S}} X_s \text{ with}$$

- (1) Each X_s is a smooth connected locally closed algebraic subvariety of X.
- (2) For all $s \in \mathcal{S}$, $\overline{X_s}$ is a union of X_t's

- (3) Technical condition ("existence of stratified slices")

c.f. [CG Def 32.23].

k commutative Noetherian ring of finite global dimension (e.g. field, $k = \mathbb{Z}$)

Rk. Assumptions ensure that

$$\mathrm{RHom}_k(-, k) : (\mathcal{D}^b\mathbf{Mod}_K^{fg})^{op} \xrightarrow{\sim} \mathcal{D}^b\mathbf{Mod}_k^{fg}.$$

$\mathbf{Sh}(X, k)$: abelian category of sheaves of k-modules on X (with respect to the **classical** topology).

If $X = \sqcup_{s\in\mathcal{S}} X_s$ is an algebraic stratification, we denote by $i_s : X_s \to X$ the embedding. Then $\mathcal{F} \in \mathcal{D}^b(\mathbf{Sh}(X, k))$ is said to be $\mathcal{S}$-*constructible* if for all $s \in \mathcal{S}$ and all $j \in \mathbb{Z}$ we have $\mathcal{H}^d(i_s^*\mathcal{F})$ is a local system (= locally constant sheaves with finitely generated stalks).

$\mathcal{D}^b_{\mathcal{S}}(X, k)$ = full subcategory of $\mathcal{D}^b\mathbf{Sh}(X, k)$ whose objects are the $\mathcal{S}$-constructible complexes. (triangulated subcategory).

Remak. The technical condition on stratification is there to ensure that $\mathcal{D}^b_{\mathcal{S}}(X, k)$ is stable under Verdier duality

$$\mathbb{D}_X = \mathbf{RHom}_k(-, \omega_X).$$

Constructible derived category $\mathcal{D}^b_c(X, k)$: full subcategory of $\mathcal{D}^b\mathbf{Sh}(X, k)$ whose objects are the complexes $\mathcal{F}$ such thta there exists an algebraic stratification $\mathcal{S}$ such that $\mathcal{F}$ is $\mathcal{S}$-cosntructible.

Again, this is a triangulated subcategory of $\mathcal{D}^b\mathbf{Sh}(X, k)$.

1.2. Operations on constructible copmlexes.

$f : X \to Y$ morphism of algebraic varieties, then we have triangulated functors

$$f_*, f_! : \mathcal{D}^b_c(X, k) \to \mathcal{D}^b_c(Y, k)$$

$$f^*, f' : \mathcal{D}^b_c(Y, k) \to \mathcal{D}^b_c(X, k)$$

Verdier duality : $\mathbb{D}_X := \mathbf{RHom}(-, \omega_{\mathbf{X}})$, where $\omega_X = \partial^! \underline{k}_{pt}$ with $\partial : X \to pt$.
Derived tensor product : $-^c \otimes_k - : \mathcal{D}^b_c(X, k) \times \mathcal{D}^b_c(X, k) \to \mathcal{D}^b_c(X, k)$.

Main properties.

- *Compatibility with convolution.*

$$(f \circ g)_* = f_* \circ g_*$$

$$(f \circ g)^* = g^* \circ f^*$$

- *Adjunctions.* (f^*, f_*) and $(f_!, f^!)$ are adjoint pairs.
- *Special cases.* If f proper then $f_* = f_!$. If f is smooth of relative dimension n then $f^! = f^*[2n]$
- *Verdier duality.* $\mathbb{D}_X \circ \mathbb{D}_X \approx \mathrm{id}$

$$\mathbb{D}_Y \circ f_* \cong f_! \circ \mathbb{D}_X,$$
$$\mathbb{D}_X \circ f^* \cong f' \circ \mathbb{D}_Y.$$

- *Base change.*

Insert cartesian square

X –f–> Y | | g' g | | v v X' –f'-> Y'

Then we have $(f')^* \circ g_! \approx (g')_! \circ f^*$ and $(f')^! \circ g_* \approx (g')_* \circ f^!$.

- *Glueing* $j : U \to X$ open embedding and $i : Z \to X$ closed embedding with $X = U \sqcup Z$.

Then

- $i_* = i_!$, $j_* = j_!$ are fully faithful
- We have functorial distinct triangles

$$j_! j^* \overset{\mathrm{adj}}{\to} \mathrm{id} \overset{\mathrm{adj}}{\to} i_* i^* \overset{[1]}{\to}$$
$$i_* i^! \overset{\mathrm{adj}}{\to} \mathrm{id} \overset{\mathrm{adj}}{\to} j_* j^* \overset{[1]}{\to}$$

1.3. Perverse sheaves (for middle perversity)

As before, k commutative Noetherian ring of finite global dimension, $X = \sqcup_{s \in \mathcal{S}} X_g$ algebraic stratification.

Def.

$${}^p\mathcal{D}^{\geq 0} = \{\mathcal{F} \in \mathcal{D}^b_{\mathcal{S}}(X,k) | \forall s \in \mathcal{S},\ i_s^! \mathcal{F} \in \mathcal{D}_c^{\geq -\dim X_s}(X_s, k)\}$$

$${}^p\mathcal{D}^{\leq 0} = \{\mathcal{F} \in \mathcal{D}^b_{\mathcal{S}}(X,k) | \forall s \in \mathcal{S},\ i_s^* \mathcal{F} \in \mathcal{D}_c^{\leq -\dim X_s}(X_s, k)\}$$

Define $\mathbf{Perv}_{\mathcal{S}}(X,k) = {}^p\mathcal{D}^{\geq 0} \cap {}^p\mathcal{D}^{\leq 0}$.

Theorem. $({}^p\mathcal{D}^{\geq 0}, {}^p\mathcal{D}^{\leq 0})$ is a bounded t-structure on $\mathcal{D}^b_{\mathcal{S}}(X,k)$. In particular, $\mathbf{Perv}_{\mathcal{S}}(X,k)$ is an abelian category, and the exact sequences in $\mathbf{Perv}_{\mathcal{S}}(K,k)$ are obtained in distinct triangles in $\mathcal{D}^b_{\mathcal{S}}(X,k)$ all of whose vertices belong to $\mathbf{Perv}(X,k)$ by forgetting the last arrow

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H} \overset{[1]}{\to},$$

where $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are perverse.

Very useful fact. For $\mathcal{G}, \mathcal{G} \in \mathbf{Perv}_{\mathcal{S}}(X, k)$

$$\mathrm{Ext}^1_{\mathbf{Perv}_{\mathcal{S}}(X,k)}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}^b_{\mathcal{S}}(X,k)}(\mathcal{F}, \mathcal{G}[1])$$

This is **not** true for highter Ext's

Intersection cohomology complexes. $s \in \mathcal{S}$, d local systemon X_s.

Claim. There exists a unique object $IC(X, \mathcal{L}) \in \mathbf{Perv}_{\mathcal{S}}(X, k)$ such that

- $IC(X_s, \mathcal{L})$ is supported on $\overline{X_s}$
- $s^* IC(X_s, \mathcal{L}) = \mathcal{L}[\dim X_s]$
- For all $t \in \mathcal{S}$ such that $X_t \subset \overline{X_s}$ and $t \neq s$ we have

$$i_t^* IC(X_s, \mathcal{L}) \in D^{<-\dim(X_t)}(X_t, k)$$
$$i_t^! IC(X_s, \mathcal{L}) \in D_c^{>-\dim(X_t)}(X_t, k).$$

We have natural maps

$${}^p\mathcal{H}^0(i_{s!}\mathcal{L}[\dim X_s]) \to {}^p\mathcal{H}^0(i_{s*}\mathcal{L}[\dim X_s])$$

factoring through $IC(X_s, \mathcal{L})$ by surjection, then injection.

Theorem. Assume that k is a field. We have a bijection

$$\{(s, \mathcal{L}) | s \in \mathcal{S} \; \mathcal{L} \text{ simple local system on } X_s\}/\text{isom} \xrightarrow{\sim} \{\text{simple objects in } \mathbf{Perv}_{\mathcal{S}}(X, k)\}/\text{isom}$$
$$(s, \mathcal{L}) \mapsto IC(X_s, \mathcal{L}).$$

Remark. If k is a field, then $\mathbb{D}_X IC(X_s, \mathcal{L}) \approx IC(X_s, \mathcal{L}^\vee)$. In particular, $\mathbb{D}_X$ restricts to an equivalence

$$\mathbf{Perv}_{\mathcal{S}}(X, k)^{op} \xrightarrow{\sim} \mathbf{Perv}_{\mathcal{S}}(X, k).$$

This is **not** true for general coefficient (already for X =point).

Ex. If $\overline{X_s}$ is smooth then $IC(X_s, \underline{k}) \approx \underline{k}_{\overline{X_s}}[\dim X_s]$.

1.4. Stratified semismallness

$X = \sqcup_{s \in \mathcal{S}} X_s$, $Y = \sqcup_{t \in \mathcal{T}} Y_t$ algebraic variety with algebraic stratification, $f : Y \to X$ **proper** such that

- (1) For all $t \in \mathcal{T}$, $f(Y_t)$ is a union of strata.

- (2) For all $t \in \mathcal{T}$ such that $X_S \subset f(Y_t)$ for all $x \in X_s$ we have

$$\dim(f^{-1}(X_s) \cap Y_t) \leq \frac{1}{2}(\dim Y_t - \dim X_s)$$

- (3) For all $t \in \mathcal{T}$ for all $s \in \mathcal{S}$ such that $X_s \subset f(Y_t)$ then the map $Y_t \cap f^{-1}(X_s) \to X_s$ induced by f is a Zariski locally trivial fibration.

Proposition. In this setting, if $f \in \mathbf{Perv}_{\mathcal{T}}(Y, k)$ then $f_*\mathcal{F} = f_!\mathcal{F}$ belongs to $\mathbf{Perv}_{\mathcal{S}}(X, k)$.

1.5. Equivariant perverse sheaves

$X = \sqcup_{s \in \mathcal{S}} X_s$ algebraic variety with algebraic stratification.

H connected complex algebraic group acting on X with each X_s H-stable we have two maps

$$H \times X \underset{p \text{ projection}}{\overset{a \text{ action}}{\longrightarrow}} X.$$

Def. $\mathcal{F} \in \mathbf{Perv}_{\mathcal{S}}(X, k)$ is H-equivariant if $a^*\mathcal{F} \xrightarrow{\sim} p^*\mathcal{F}$.

$\mathbf{Perv}_{\mathcal{S},H}(X, k)$ full subcat of $\mathbf{Perv}_{\mathcal{S}}(X, k)$ whose objects are H-equivariant perverse sheaves.

Facts.

- (1) $\mathbf{Perv}_{\mathcal{S},H}$ is an abelian subcategory, stable under subquotients (but not under extensions in general).
- (2) If $\mathcal{L}$ H-equivariant local system on X_s, then $IC(X_s, \mathcal{L})$ is H-equivariant
- (3) $\mathbf{Perv}_H(X, k)$ is the heart of the perverse t-structure on the H equivariant $\mathcal{S}$-contructible derived category of Bernstein-Lunts.
- (4) If $X \to Y$ is a (Zariski locally trivial) H-torsor and $\mathcal{S}$ is the pullback of the stratification $\mathcal{T}$ on Y, then the category $\mathbf{Perv}_{\mathcal{S},H}(X, k)\overline{X} \to \to \mathbf{Perv}_{\mathcal{T}}(Y, \mathcal{T})$.

1.6. Partity complexes.

$X = \sqcup_{s \in \mathcal{S}} X_s$ algebraic variety with algebraic stratification.

Assumptions :

- k is a field
- For any $s \in \mathcal{S}$ all local systems on X_s are trivial (i.e. the fundamental groups of X_s's are trivial)

- For all $s \in \mathcal{S}$ we have $\mathrm{H}^{\mathrm{odd}}(X_s; k) = 0$.

Def. $\mathcal{F} \in \mathcal{D}^b_{\mathcal{S}}(X,k)$ is called *even* (resp. *odd*) if $\mathcal{H}^{\mathrm{odd}}(\mathcal{F}) = \mathcal{H}^{\mathrm{odd}}(\mathbb{D}_X \mathcal{F}) = 0$ (resp. $\mathcal{H}^{\mathrm{even}}(\mathcal{F}) = \mathcal{H}^{\mathrm{even}}(\mathbb{D}_X \mathcal{F}) = 0$).

$\mathcal{F}$ is called *parity* if it is a direct sum of an even object and an odd object.

Exercise. If $\mathcal{F}$ is even and $\mathcal{G}$ is odd, then

$$\mathrm{Hom}_{\mathcal{D}^b_{\mathcal{S}}(X,k)}(\mathcal{F}, \mathcal{G}) = 0.$$

Theorem. (Juteau – Mautner– Williamson) For any $s \in \mathcal{S}$ there exists at most one indecomposable object parity complex $\mathcal{E}_s \in \mathcal{D}^b_{\mathcal{S}}(X,k)$ supported on $\overline{X_s}$ and such that $i^*_{S}\mathcal{E}_s \xrightarrow{\sim} \underline{k}[\dim X_s]$. Moreover, any indecomposable parity complex is of the form $\mathcal{E}_f[n]$ for some $s \in \mathcal{S}$ and $n \in \mathbb{Z}$, and any parity complex is a direct sum of indecomposable parity complexes.

Remark. It can happen that $\mathcal{E}_s$ does not exist. But it always exists for X affine Grassmannian (with the stratification by orbits of a parahoric subgroup), or for partial flag varieties of Kac-Moody groups.