Representations of complex Lie algebras and Weyl character formula. Part 2.

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1. Characters.

Let $\mathbb{Z}[\Lambda]$ be a group ring on Λ. Explicitely, $\mathbb{Z}[\Lambda] = \{\sum_{i=1}^{k} a_i x^{\lambda_i} : x^{\lambda} \leftrightarrow \lambda \in \Lambda\}$. Product $x^{\lambda} x^{\mu} = x^{\lambda+\mu}$.

Define $R(\mathfrak{g})$ be the representation ring with elements being isomorphism classes of $\mathfrak{g}$-representations, with ring stucture given by direct sum and tensor product.

Define $\text{char} : R(\mathfrak{g}) \to \mathbb{Z}[\Lambda]$ by $\text{char}(V) = \sum \dim(V_\lambda) e(\lambda)$ where $V = \oplus_\lambda V_\lambda$.

Theorem. $R(\mathfrak{g})$ is a polynomial ring with variables $\Gamma_{w_1}, \Gamma_{w_2}, \cdots, \Gamma_{w_n}$.

$$R(\mathfrak{g}) \sim \mathbb{Z}[\Lambda]^W \text{ where } W = \text{ Weyl group.}$$

Back to $\mathfrak{sl}_n(\mathbb{C}) : V = \mathbb{C}^n$.

Recall : Weyl construction : $\mathbb{S}_\lambda V = V^{\otimes d} \otimes_{\mathbb{C}S_d} \tilde{V_\lambda},\ d = |\lambda|$.

Where $\tilde{V_\lambda}$ is the Specht module associated to λ.

e.g. $\mathbb{S}_{(d,0,\cdots,0)} V = \text{Sym}^d(V),\ \text{trSym}^d(V) = h_d = \text{char}(\mathbb{S}_{(d,0,\cdots,0)} V)$, and $\mathbb{S}_{(1,1,\cdots,1,0,\cdots,0)} V = \bigwedge^d V$, $\text{tr} \bigwedge^d V = e_d = \text{char}(\mathbb{S}_{(1,1,\cdots,1,0,\cdots,0)} V)$.

Recall : $\text{Sym}^{\lambda_1}(V) \otimes \cdots \otimes \text{Sym}^{\lambda_n}(V) \cong \bigoplus_{\mu \le \lambda} K_{\lambda\mu} \mathbb{S}_\mu V$.

By Young's rule,

$$\text{char}\left(\text{Sym}^{\lambda_1}(V) \otimes \cdots \otimes \text{Sym}^{\lambda_n}(V)\right) = \text{char}(\text{Sym}^{\lambda_1} V) \cdots \text{char}(\text{Sym}^{\lambda_n} V) = h_{\lambda_1} \cdots h_{\lambda_n} = h_\lambda = \sum_{\mu \le \lambda} K_{\lambda\mu} s_\mu.$$

But also

$$\text{char}\left(\text{Sym}^{\lambda_1}(V) \otimes \cdots \otimes \text{Sym}^{\lambda_n}(V)\right) = \sum_{\mu \le \lambda} K_{\lambda\mu} \text{char}(\mathbb{S}_\mu V),$$

which yields $\text{char}(\mathbb{S}_\mu V) = s_\mu$.

$\dim \mathbb{S}_\mu V = s_\mu(1,1,\cdots,1) = \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j-i}$.

Prop. $\mathbb{S}_\lambda V$ is irrep of $\mathfrak{sl}_n\mathbb{C}$, with highest weight $\sum \lambda_i L_i$

So $\Gamma_{a_1,\cdots,a_{n-1}} \leftrightarrow \mathbb{S}_{(a_1+\cdots+a_{n-1}, a_2+\cdots+a_{n-1},\cdots,a_{n-1},0)}V$.

Weyl Character formula :

In $\mathfrak{sl}_n\mathbb{C}$, $\text{char}(\mathbb{S}_\lambda V) = s_\lambda = \frac{\det[x^{\lambda_i+n-i}]}{\det[x^{n-i}]}$.

Note that

$$\det[x^{\lambda_i+n-i}] = \sum_{\sigma\in S_n = W_{\mathfrak{sl}_n\mathbb{C}}} \text{sgn}(\sigma) \prod x_i^{\lambda_{\sigma(i)}+n-\sigma(i)}.$$

Write $\rho = (n-1, n-2, \cdots, 1, 0) \leftrightarrow \sum_{i=1}^{n-1} w_i = \frac{1}{2}\sum_{i<j} L_i - L_j$, therefore $\det[x^{\lambda_i+n-i}] = \sum_{\sigma\in W} \text{sgn}(\sigma) x^{\sigma(\lambda+\rho)}$.

Write $A_\lambda = \sum_{\sigma\in W} \text{sgn}(\sigma) x^{\sigma(\lambda)}$.

So $\det[x^{\lambda_i+n-i}] = A_{\lambda+\rho}$ and

$$A_\rho = \det[x^{n-i}] = \prod_{i<j}(x_i - x_j) = \underbrace{(x_1\cdots x_n)}_{=1} \prod_{i<j}\left(\frac{x_i^{1/2}}{x_j^{1/2}} - \frac{x_j^{1/2}}{x_i^{1/2}}\right) = \prod_{i<j}\left(\underbrace{\frac{x_i^{1/2}}{x_j^{1/2}}}_{x^{1/2L_i - 1/2L_j}} - \underbrace{\frac{x_j^{1/2}}{x_i^{1/2}}}_{x^{1/2L_j - 1/2L_i}}\right).$$

2. Weyl character formula :

$\text{char}(\Lambda_\mu) = \frac{A_{\lambda+\rho}}{A_\rho}$.

Fact. $A_\rho = \sum_{\sigma\in W} \text{sgn} x^{\sigma(\lambda)} = \prod_{\alpha\in R^+}(x^{\alpha/2} - x^{-\alpha/2}) = x^\rho \prod_{\alpha\in R^+}(1 - e^{-\alpha}) = x^{-\rho}\prod_{\alpha\in R^+}(e^\alpha - 1)$.

Cor. $\dim \Gamma_\lambda = \prod_{\alpha\in R^+} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}$.

Naively, $\dim \Gamma_\lambda = \frac{A_{\lambda+\rho}(1,1,\cdots,1)}{A_\rho(1,\cdots,1)} = \frac{S_{\lambda+\rho}(1,\cdots,1)}{S_\rho(1,1,\cdots,1)}$ (in $\mathfrak{sl}_n\mathbb{C}$).

So $\Psi_\rho(A_\lambda) = \Psi_\lambda(A_\rho) = \prod_{\alpha\in R^+}(e^{(\lambda,\alpha)/2t} - e^{-(\lambda,\alpha)/2t})$.

Expand in terms of t, we $\prod_\alpha \in R^+((\lambda,\alpha)t +$ higher powers ot $t) = (\prod_{\alpha\in R^+(\lambda,\alpha)}(\lambda,\alpha))t^{|R^+} + \sum$ higher powers of t.

So

$$\Psi_\rho(\text{char}\Gamma_\lambda) = \frac{\Psi(A_{\lambda+\rho})}{\Psi_\rho(A_\rho)} = \frac{\prod_{\alpha\in R^+}(\lambda+p,\alpha)}{\prod_{\alpha\in R^+}(\rho,\alpha)} + \text{ higher powers.}$$

Then one can set $t=0$ and get the desired formula.