Satake, Weyl character formula, MacDonald summary : Part 2

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F local non-archimedean field. $\mu \in \mathbf{X}_\star(T) = \mathbf{X}^\star(\hat{T})$, f_μ characteristic function of $K(\varpi^\mu)K$, is a function on $G(F)$.

If $\mu > 0$ then τ_μ = characteristic function of V_μ, representation of $\hat{G}(\mathbb{C})$ of highest weight μ.

Spherical representations of G are in 1-1 correspondance to $\mathbb{C}[\hat{T}]^W$.

For $\lambda \in \mathbf{X}^\star(\hat{T})$, let e^λ be a formal symbol.

$$\mathbb{C}[\mathbf{X}^\star(\hat{T})] \cong \mathbb{C}[\hat{T}],$$

via $\sum c_\lambda \cdot \lambda = \sum c_\lambda e^\lambda \mapsto \sum c_\lambda \lambda(t)$.

Natural bases for $\mathbb{C}[\mathbf{X}^\star(\hat{T})]^W$:

- 1. $\sum_{w \in W/W(\mu)} e^{w\mu}$ (one for each μ).
- 2. $S(f_\mu)$ (Satake transform).
- 3. $\{\tau_\mu\}_{\mu > 0}$.

Macdonald : rewrite $\{S(f_\mu)\}$ in terms of $\{\tau_\lambda\}$.

$\delta^{-1/2}(\varpi^\lambda) = q^{\langle \lambda, f^\vee \rangle}$

Macdonald: $S(f_\mu) = \mathrm{vol}(M_\mu) q^{\langle \mu, \rho^\vee \rangle} \left(\sum_{w \in W} \frac{\prod_{\alpha>0}(1-q^{-1}e^{-w\alpha})}{\prod_{\alpha>0}(1-e^{-w\alpha})} \cdot e^{w\mu} \right)$.$

Weyl's character formula. $\lambda > 0$, $\tau_\lambda = \sum_{w \in W} \frac{e^{w\lambda}}{\prod_{\alpha>0}(1-e^{-w\alpha})}$.

Expand the numerator of Macdonald's formula, and invert the order of the sums.

$$= \mathrm{vol}(M_\mu) q^{\langle \mu, f^\vee \rangle} \sum_{S \subset \Phi^+} \sum_{w \in W} \frac{e^{w(\mu - \alpha_S)}}{\prod_{\alpha>0}(1 - e^{-w\alpha})},$$

where $\alpha_S = \sum_{\alpha \in S} \alpha$.

$$= \mathrm{vol}(M_\mu) q^{\langle \mu, \rho^\vee \rangle} \sum_{S \subset \Phi^+} (-q)^{|S|} \tau_{\mu - \alpha_S}.$$

This is not quite right, because Weyl character formula is for dominant weights. While μ is dominant, $\mu - \alpha_S$ doesn't have to be.

Trick 1. $\frac{1}{1-e^{-w\alpha}} = 1 + e^{-w\alpha} + e^{-2w\alpha} + \cdots$, plug those in formula and invert sums. We saw in Nick's talk that we can rewrite Weyl character formula by $\tau_\lambda = \frac{\sum_W \mathrm{sgn}(w) e^{w(\lambda+\rho)}}{\sum_W \mathrm{sgn}(w) e^{w\rho}}$., this works for all λ.

Macdonad's formula becomes :

$$S(f_\lambda) = \mathrm{vol}(M_\lambda) \cdot f^{\langle \lambda, f^\vee \rangle} \sum_{S \subset \Phi^+} (-q)^{|S|} \prod_{\lambda + (\rho - \alpha_S)}.$$

Let $\rho_S = \rho - \alpha_S$, so $\rho_\varnothing = \rho$, $\rho_{\Phi^+} = -\rho$. Every $w \cdot \rho$ is one of the ρ_S (S = set of roots participating in the expression for w).

Let $C_\rho = \{\rho_S | S \subset \Phi^+\}$, set of all weights of the irrep V_ρ of highest weight ρ.

For each $\mu \in C_\rho$, define $P_\mu(x) = \sum_{S : f_S = \mu} x^{|S|}$. (so $P_\rho = 1$, $R_w \rho(x) = x^{\ell(x)}$), set $x = 1 : P_\mu(1)$ = multiplicity of the weight μ in V_ρ.

Get: $S(f_\mu) =$
$mathrmvol(M_\lambda) q^{\langle \lambda, \rho^\vee} \sum_{\mu \in C_\rho} P_\mu(-q^{-1}) \Pi_{\lambda+\mu}$.

Conversely, $\tau_\lambda = \sum_{\mu \text{ dominant}, \, \mu \leq \lambda} K_{\mu,\lambda}(q^{-1}) S(f_\mu)$, up to the factor $q^{\langle \lambda - \mu, \rho^\vee \rangle}$, these are the Kazhdan-Lustig polynomials for (μ, λ).

Note. $K_{\mu,\lambda}$ are associated with the affine Grassmannian "$G(F)/K$". Start with a full coxeter group : affine Weyl group.

Schur polynomials corresponds to something, and Weyl group corresponds to G/B.

Back to Weyl Character formula.

We go over theorem from **Casselman–Cely–Hales.**

Let V a finite-dimensional representation of $\hat{G}(\mathbb{C})$, with $V = \bigoplus V_\mu$ its weight space decomposition. $E : V \to V$ formal linear operator.

Our E will be diagonal with respect to $V = \bigoplus V_\mu$ on each V_μ it is mult by e^μ. We have $\det(1 - qE, V) \in \mathbb{Z}[e^\mu, q]$. Define $P(\tilde{G}, V, E, q) = \det(1 - qE, V)^{-1}$, this is called a *q-partition function.*

- For $q = 1$,
$$\det(1 - E, \text{ adj. rep on } \mathfrak{g}/\mathfrak{z} = \prod_{\alpha \in P} (1 - e^{-\alpha}).$$

$\prod_{\alpha>0}(1-e^{-\alpha})^{-1} = \sum e^{-\mu}$·Kostant partition function μ. (Usin the trick $\prod(1-e^{-\alpha})^{-1} = 1 + e^{-\alpha} + e^{-2\alpha} + \cdots$).

Remark. G acts on $\mathrm{Sym}(\mathfrak{g})$, which is a free module over $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{g}]^W$. Harmonic polynomials on $\mathfrak{g}$ are killed by invariant constant coefficient differential operators (elements of $\mathrm{Sym}(\mathfrak{g}^\star)$) of positive degree. Call those polynomials $\mathcal{H}(\mathfrak{g})$. So we get $\mathrm{Sym}(\mathfrak{g}) = \mathcal{H}(\mathfrak{g}) \otimes \mathbb{C}[\mathfrak{g}]^G$, and G acts on the left part. Multiplicities of G On $\mathcal{H}(\mathfrak{g})$ are finite. μ-integral dominant weight is a element of Λ, the root lattice. So V_μ occurs with q-multiplicity. $q^{\deg}$ (general exponents of μ) are multiplicity of μ in the corresponding graded piece.

Weyl character formula corresponds (by expanding denominator) to Kostant multiplicity formula (multiplicity of μ in V_λ)

$$\mathrm{mult}(\mu) = \sum_{w\in W} (-1)^{\ell(w)} P(w\cdot(\lambda+\rho) - \mu + \rho),$$

where P is the Kostant partition function.

Side note: Take λ dominant weight. $Z_\lambda = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$ (Verma module), here C_λ is a 1 dimensional representation of $\mathfrak{b} : \mathfrak{g}$ acts by λ. The multiplicity of μ in Z_λ is $P(\lambda-\mu)$ (number of ways to get from λ to μ by using negatice roots).

Weyl denominator $q-$determinant with $q=1$, $V=\pi$ adjoint representation.

$\mathfrak{g} = \pi^{-1} + \mathfrak{g} + \underbrace{\pi}_{\oplus_{\alpha>0}\mathfrak{g}_\alpha}$.

Sketch of the proof of Weyl character formula: $\tau_\lambda = J(e^\lambda)P(E^{-1},1)$, J is the *Weyl symmetrizer operator*, $J(f)(x) = \sum_W (-1)^{\ell(w)} f(w\cdot x)$.

Make a chain complex: $C^j = \bigwedge^j \pi' \otimes V_\lambda$. $T_\lambda =$ character on the cohomology of this complex.

We had : coeff of τ_λ in terms of $S(f_\mu)$ ($\langle \tau_\lambda, S(f_\mu)\rangle_{\mu^{\mathrm{Pl}}_{\hat{T}(\mathbb{C})}}$).

Fact : Pl measure on $\hat{T}$: $\frac{P(E,q^{-1})P(E^{-1},q^{-1})}{P(E,1)P(E^{-1},1)}\mathrm{d}s$. If $s\in\hat{T}$, E depends on $s = q^{2\pi i\lambda}$, $E = E_\lambda$