Satake, Weyl character formula, MacDonald summary: Part 1.

Julia Gordon

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F is a p-adic field.

G is a split connected reductive group over F. $G = G(F)$

"Spherical part of the spectrum of $G(F)$".

Analogy. Fourier transform on $\mathbb{R}$.

From f, a function on $\mathbb{R}$ (Schwarz, ...) we get $\hat{f}$ on $\hat{\mathbb{R}} \cong \mathbb{R}$ (the isomorphism isn't canonical), where $\hat{\mathbb{R}}$ is $\{\psi_x : \psi_x(y) = e^{2\pi i x y} | x \in \mathbb{R}\}$

$$\text{(Fourier inversion) } f(x) = \text{(constant)} \int_{\hat{\mathbb{R}}} \hat{f}(\Psi)\overline{\Psi(x)}\, \mathrm{d}\psi.$$

Important: $\mathrm{d}\psi$ is a measure on $\hat{\mathbb{R}}$, what allows us to compute it is that $\mathrm{d}x = \mathrm{d}\psi$.

$\hat{f}(\psi_x) =$ "Fourier coefficient of f at x"

Our group is $G(F)$, the analogy gives all (unitary?) reps of $G(F)$, too complicated. If we look at the smaller subset of spherical representations, i.e. π having a K-fixed vector ($K = G(\mathcal{O}_F)$). This smaller subset has a 1-1 correspondence with bi-K-invariant functions, which we expect to be "orthogonal" to the part of the spectrum without fixed vectors. (Not quite true, but true replacing K by I)

What is true : We can recover such a function f from its "Fourier coefficients" at spherical representations (= Satake transform of f).

$f \in \mathcal{H}(G//K) \to S(f)(\pi) = \int_G f(g)(\text{matrix coeff of } \pi)\mathrm{d}g =$ "$\hat{f}(\pi)$".

Know : Spherical representations : $\pi = \mathrm{ind}_B^G(|\cdot|_F^{s_1} \times \cdots \times |\cdot|_F^{s_n})$, $s_i \in \mathbb{C}$. Here $\mathrm{ind} = \mathrm{Ind}() \otimes \delta_B^{-1/2}$. When this is reducible, has a unique spherical subquotient (by rearranging $\{s_i\}$, can force it to be a quotient).

Let $(z_i = |\varpi|^{s_i}) \in (\mathbb{C}^\times)^n/W$, identify π as $\pi_{(z_1,\ldots,z_n)}$.

For Fourier transform of bi-K-invariant function can be found just with the spherical part of representations via Satake transform, because it's an isomorphism, don't need the rest of reps !

So $S(f)$ is a function on $(\mathbb{C}^\times)^n/W \leftrightarrow W$-invariant functions on $(\mathbb{C}^\times)^n$, it is actually regular ! (i.e. W-invariant polynomial in $z_i^{\pm 1}$).

We got our formula $S(f)(\pi_z) = \int_G f(g)E_z(g)\ g$

$z \in (\mathbb{C}^\times)^n/W$, $E_z(g)$ is a bi-K-invariant matrix coefficient such that $E_z(1) = 1$, it is a spherical function.

If we believe that $S(f)$ is a polynomial in $z_i^{\pm 1}$, get $S(f) \in \mathbb{C}[\hat{T}/W]$, $\hat{T} =$ Langlands dual torus of $T \subset G$ i.e. $\mathbf{X}^\star(\hat{T}) = \mathbf{X}_\star(T)$.

$T(\mathbb{C}) = \mathbf{X}_\star(\hat{T}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbf{X}^\star(T) \otimes \mathbb{C}$.

Relation to Langlands

(packets of) representations of $G(F)$ have a correspondence with Langlands parameters σ : Weyl-Deligne group of $F \to {}^L G$. For spherical : "Deligne" is not relevant. Weil group of F.

$$1 \to I_F \to \underset{W_F \subset}{\mathrm{Gal}(\overline{F}/F)} \to \langle \mathrm{Frob} \rangle \to 1.$$

The term on the right is the Galois group of a maximal unramified extension. "Unramified" here can be seen as $\sigma|_{I_F} = 1$.

$T \subset G \leftrightarrow (\mathbf{X}^\star(T), \phi, \mathbf{X}_\star(T), \phi^\vee)$. The group ${}^L G(\mathbb{C})$ is a complex Lie group with max torus $\hat{T}$, root system $\phi^\vee$. $\sigma(\mathrm{Frob}) \in \hat{T}(\mathbb{C})/W$.

Theorem. Satake transform is an isomorphism $\mathcal{H}(G//K) \to \mathbb{C}[\hat{T}/W]$.

Thomas proved surjectivity.

Injectivity : closely related to "Plancherel formula".(

On G : $(f_1, f_2) = \int_G f_1(g)\overline{f_2(g)}\mathrm{d}g = \int_{R(G)} \hat{f}_1 \overbrace{f_2^\vee}^{=\hat{f}_2(\tilde{\pi})} \mathrm{d}\pi$ ($\tilde{\pi}$ is the contragradient rep). Plancherel measure on $\mathbf{R}(G) =$ space of unitary reps of G.

The Plancherel measure on $\hat{T}(\mathbb{C})/W$ would give $f_{1,2} \in \mathcal{H}(G//K)$, $(f_1, f_2) = \int_{\hat{T}/W} S(f_1)\overline{S(f_2)}\ \mathrm{d}\mu^{\mathrm{Pl}}(z_1, \cdots, z_n)$.

Question. Plancherel measure on $\hat{T}(\mathbb{C})/W$?

The algebra $\mathbb{C}[\hat{T}/W] \cong \mathbb{C}[\underbrace{\mathbf{X}^\star(\hat{T})}_{\text{lattice}}]$:

- Natural generators (as an algebra):

$\xi_1, \cdots, \xi_n$ = "elementary symmetric polynomials" (they can be seen as the characters of irreducible representations corresponding to the fundamental weights).

- Bases as $\mathbb{C}$-vector space (= W-invariant polynomial functions on $\hat{T}(\mathbb{C}) = (\mathbb{C}^\times)^n$).

One natural basis : $\mu \in \mathbf{X}^\star(\hat{T}) \to e^\mu$, a formal symbol. Elements of $\mathbb{C}[\mathbf{X}^\star(\hat{T})]$ (convolution ring, with this notation it is product of polynomials) are of the form $\sum_\mu c_\mu e^\mu : t \mapsto \sum c_\mu \mu(t)$ (finitely many nonzero terms).

- The first natural basis for $\mathbb{C}[\mathbf{X}^\star(T)]^W$ is $\sum_{\mu \in W(\lambda)} e^\mu$, where $W(\lambda)$ is the orbit of λ under W.
- The second basis : τ_λ = character of an irreducible representation of ${}^L G$ of highest weight.
- $S(f_\mu)$, f_μ character function of $K \begin{pmatrix} \varpi^{\mu_1} & & 0 \\ & \ddots & \\ 0 & & \varpi^{\mu_n} \end{pmatrix} K$ (double coset corresponding to $\mu = (\mu_1, \cdots, \mu_n) \in \mathbf{X}_\star(T) = \mathbf{X}^\star(\hat{T})$).

Macdonald's formula gives a way to give a change of basis from $\{S(f_\mu)\}$ to $\{\tau_\mu\}$.