Macdonald's formula. Part 2.

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Let $\int = \{z \in (\mathbb{C}^\times)^d : |\chi_z(\alpha^\vee(\varpi_F)) < 1 \ \forall \alpha \in R_+\}$.

Recall A is a maximal F-split torus with A^+ its positive part, $a^{j_1} \cdots a^{j_d}$, $j_1 \geq \cdots \geq j_d$.

W is the Weyl group with longest element w_0, with fixed representatives in K. I is the Iwahori subgroup.

Recall the term appearing in Satake transform

$$E_z(a) = \int_K f_{z.B}(ka) \ \mathrm{d}k = \sum_{w \in W} \mathrm{d}(w,k)\delta_B^{1/2}(A)\chi_{wz}(a)$$

, $a \in A^+$.

Recall also the proposition proved last time

Proposition. Let $z \in \int$ and let $q(w) = [IwI : I]$ and put $Q = \sum_{w \in W} q(w)$. Then

$$\mathrm{d}(w_0, z) = Q^{-1} \int_{w_0 N w_0^{-1}} f_{z,B}(n)\mathrm{d}n.$$

where $f_{z,B}(a) = (\delta_B^{1/2} \cdot \chi_z)(a)$

Proof. By the corollary

$$E_z(a) = \sum_{k \in I \backslash K} \int_I f_{z,B}(ika) \frac{\mathrm{d}i}{[K:I]}.$$

FACT : One was $f_{z,B}(iwa) = f_{z,B}(iwaw^{-1}) = f_{z,B}(iwa)(iw(a)) = f_{z,B}(w(a))f_{z,B}(w(a)^{-1}iw(a))$.

Thus

$$E_z(a) = \frac{1}{[K:I]} \sum_{w \in W} c(w) \int_I f_{z,B}(iwa)\mathrm{d}i$$

where $c(w) = [Iw(N^- \cap K) : I]$.

Indeed,

$$\int_I f_{z,B}(w(a)^{-1}iw(a))\mathrm{d}i = \int_{N\cap I}\int_{A\cap I}\int_{N^-\cap I} f_{z,B}(w(a)^{-1}n^+xn^-w(a))\mathrm{d}n^-\mathrm{d}x\mathrm{d}x^+.$$

$$= \int_{N\cap I}\int_{A\cap I}\int_{N^-\cap I} f_{z,B}(w(a)^{-1}n^+w(a)\underbrace{w(a)^{-1}xw(a)}_{\in K}\underbrace{w(a)^{-1}n^-w(a)}_{\in K})\mathrm{d}n^-\mathrm{d}x\mathrm{d}x^+.$$

Rewriting $E_z(a)$, we have $E_z(a) = \frac{1}{[K:I]}\sum_{w\in W} c(w)(\delta_B^{1/2}\chi_z(w(a)))\int_{I\cap N} f_{z,B}(w(a)^{-1}xw(a))\mathrm{d}a.$

Put $J_{w(a)} = (N\cap wa(I\cap N)a^{-1}w^{-1})\backslash(wa(I\cap N)a^{-1}w^{-1})$.

FACTS :

- If $a\in A^+$ and $w\neq w_0$ then $\chi_{w_0z}(a)^{-1}\chi_z(w(a))$ is a "decreasing exponential" as $a\to\infty$. This implies that $\mathrm{d}(w_0,z) = \lim_{\substack{a\in A^+\\ a\to\infty}}\delta_B^{1/2}(a)\chi_{w_0z}(a)^{-1}E_z(a)$,

$$= \lim_{\substack{a\in A^+\\ a\to\infty}}\delta_B^{1/2}\chi_{w_0z}(a)^{-1}\frac{1}{[K:I]}\sum_{w\in W} c(w)(\delta_B^{1/2}\cdot\chi_z)(w(a))\int_{J_{w(a)}} f_{z,B}(x)\ \mathrm{d}x$$

$$= \frac{1}{[K:I]}\lim_{\substack{a\in A^+\\ a\to\infty}}\sum_{w\in W} c(w)(\chi_{w_0z}(a)^{-1}\chi_z(w(a))\int_{J_{w(a)}} f_{z,B}(a)\mathrm{d}x.$$

Note that $I_{w_0}(N^-\cap K) = I_{w_0}(w_0Iw_0^{-1}) = Iw_0$ $(w_0^2=1)$ so $c(w_0)=1$.

So

$$\mathrm{d}(w_0,z) = \frac{1}{[K:I]}\lim_{\substack{a\in A^+\\ a\to\infty}}\int_{J_{w_0(a)}} f_{z,B}(x)\mathrm{d}x.$$

Observe that $(w_0a)N(w_0a)^{-1} = w0aNa^{-1}w_0^{-1} = w_0Nw_0^{-1} = N^-$.

So $N\cap(w_0a(I\cap N)a^{-1}w_0^{-1}) = \{1\}$ hence $J_{w(a)} = w_0a(I\cap N)a^{-1}w_0^{-1}$.

As a goes to ∞, the conjugate $w_0a(I\cap N)a^{-1}w_0^{-1}$ expand to fill out N^-.

That is $\lim_{\substack{a\in A^+\\ a\to\infty}} J_{w_0(a)} = w_0Nw_0^{-1} = N^-$.

Thus

$$\mathrm{d}(w_0,z) = \frac{1}{[K:I]}\int_{w_0Nw_0^{-1}} f_{z,B}(x)\ \mathrm{d}x = \frac{1}{Q}\int_{w_0Nw_0^{-1}} f_{z,B}(x)\ \mathrm{d}x$$

$[K:I] = \sum_{w\in W} q(w)$ where $q(w) = [IwI:I]$.

Theorem (Macdonald). Let $\mu\in\mathbf{X}_\star(A)^+$, then

$$f_\mu^\vee(z) = Q^{-1}\delta_B(a_\mu)^{-1/2}\sum_{w\in W}\chi_{wz}(a_{mu})\prod_{\alpha>0}\frac{1-q^{-1+\langle\alpha^\vee,w(z)\rangle}}{1-q^{\langle\alpha^\vee,w(z)\rangle}}.$$

Proof. $B' = AN^-$. From Thomas's talk :

$$\int_{w_0Nw_0^{-1}} f_{z,B}(n) = I_{B',B}f_{z,B}(1) = \prod_{\alpha>0}\frac{1-q^{-(1+\langle\alpha^\vee,z\rangle)}}{1-q^{\langle\alpha^\vee,z\rangle}}f_{z,B}(1) = \prod_{\alpha>0}\frac{1-q^{-(1+\langle\alpha^\vee,z\rangle)}}{1-q^{\langle\alpha^\vee,z\rangle}}.$$

Thus $\mathrm{d}(w_0,z) = \frac{1}{Q}\prod_{\alpha>0}\frac{1-q^{-(1+\langle\alpha^\vee,z\rangle)}}{1-q^{-\langle\alpha^\vee,z\rangle}}$

From the computation for $E_z(a)$ we have $\mathrm{d}(w,z) = \mathrm{d}(1,w(z))$ for $w\in W$.

So we now have a formula for all $\mathrm{d}(w,z)$ and thus also all $E_z(a)$ and all $f_\mu^\vee$.

Last piece : $m(Ka_\mu K) = \delta_B(a_\mu)^{-1}$.