Macdonald's formula. Part 1.

Ed Belk

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The setting, as in the talks about intertwining operators, we have $G = ANK$, where A is a maximal split F-torus, and $W = N_G(A)/A$ is the corresponding Weyl group.

Let I be the corresponding Iwahori subgroup of K (maximal compact subgroup), together with its Iwahori decomposition $I = (N \cap I)(A \cap I)(N^- \cap I) = (N^- \cap I)(A \cap I)(N \cap I)$ (also, Bruhat decomposition and $A \cap I = A \cap K$, $N \cap I = N \cap K$).

Let w_0 be the longest element of W. It satisfies $w_0 N w_0^{-1} = N^-$.

SubLemma. For all $w \in W$.

$$I_w(w_0^{-1} I w_0) = I_w(N^- \cap K).$$

Proof. One has

$w_0^{-1} I w_0 = w_0^{-1}(N^- \cap I)(A \cap I)(N \cap I)w_0 = w_0^{-1}(N^- \cap I)w_0(A \cap I)w_0^{-1}(N \cap I)w_0$, since w_0 normalizes $A \cap I$.

We get $w_0^{-1} I w_0 = w_0^{-1}(N^- \cap I)w_0(A \cap I)(N^- \cap K)$.

This $Iw(w_0^{-1} I w_0) = Iw(\underbrace{w_0^{-1}(N^{-1} \cap I)w_0}_{\subset K_1}(A \cap I)(N^- \cap K))$, where $K_1 = \{x \in G : x - 1 \text{ has only } \mathfrak{P}_F \text{ entries}\}$.

This is equal to $IwK_1(A \cap I)(N^- \cap K) = IK_1 w(A \cap I)(N^- \cap K) = I_2(A \cap I)(N^- \cap K) = I(A \cap I)w(N^- \cap K) = Iw(N^- \cap K)$.

Corollary. There is a correspondence

$$W \leftrightarrow I \backslash K/(N^- \cap K).$$

Proof. $K = Kw_0 = \bigcup_{w \in W} IwIw_0$ by Bruhat decomp. $= \bigcup_{w \in W} I_{ww_0} w_0^{-1} I w_0 = \bigcup_{w \in W} I_w(w_0^{-1} I w_0) = \bigcup_{w \in W} I_w(N^- \cap K))$.