MATH 527/427 HOMEWORK 3

(1) (a) Suppose given a commutative diagram of R-modules

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1} \arrow[r, "h_{n+1}"] \arrow[d, "\gamma_{n+1}"] & A_n \arrow[r, "f_n"] \arrow[d, "\alpha_n", "\cong"'] & B_n \arrow[r, "g_n"] \arrow[d, "\beta_n"] & C_n \arrow[r, "h_n"] \arrow[d, "\gamma_n"] & A_{n-1} \arrow[r] \arrow[d, "\alpha_{n-1}", "\cong"'] & \dots \\
\dots \arrow[r] & C'_{n+1} \arrow[r, "h'_{n+1}"] & A'_n \arrow[r, "f'_n"] & B'_n \arrow[r, "g'_n"] & C'_n \arrow[r, "h'_n"] & A'_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

in which the maps α are isomorphisms and in which the rows are long exact sequences. Define maps $\partial : C'_n \to B_{n-1}$ by $\partial = f_{n-1} \circ \alpha_{n-1}^{-1} \circ h'_n$
Prove that the sequence

$$\dots \longrightarrow B_n \xrightarrow{\beta_n, g_n} B'_n \oplus C_n \xrightarrow{g'_n - \gamma_n} C'_n \xrightarrow{\partial} A_{n-1} \longrightarrow \dots$$

is exact.

(b) Suppose A and B are two subcomplexes of a CW complex X such that $X = A \cup B$. You may use the fact $X/A = B/(A \cap B)$. Use the previous part of this question to establish the *Mayer–Vietoris* long exact sequence:

$$\longrightarrow \mathrm{H}_n(A \cap B; R) \longrightarrow \mathrm{H}_n(A; R) \oplus \mathrm{H}_n(B; R) \longrightarrow \mathrm{H}_n(X; R) \longrightarrow \mathrm{H}_{n-1}(A \cap B; R) \longrightarrow$$

(explaining what the maps are).

(2) Read the example on page 5, chapter 0 of Hatcher explaining how to produce the genus-g surface as a CW complex. That is, a genus-g surface M_g is obtained by identifying edges in a convex $4g$-gon in alternate pairs with reversed orientation on the edges. Calculate the homology $\mathrm{H}_*(M_g; R)$.

(3) Consider the map $f : \mathbb{R}P^2 \to S^2$ given by collapsing the 1-skeleton $\mathbb{R}P^1 \subset \mathbb{R}P^2$ to a point. Calculate $f_* : \tilde{\mathrm{H}}_*(\mathbb{R}P^2; \mathbb{Z}) \to \tilde{\mathrm{H}}_*(S^2; \mathbb{Z})$ and $f_* : \tilde{\mathrm{H}}_*(\mathbb{R}P^2; \mathbb{Z}/(2)) \to \tilde{\mathrm{H}}_*(S^2; \mathbb{Z}/(2))$

(4) (a) Show that the degree of the antipodal $a : S^n \to S^n$ map given by $a(v) = -v$ is $(-1)^{n+1}$.

(b) Show that if $f : S^n \to S^n$ is a map of spheres without a fixed point, that $f \simeq a$.

(c) Suppose $g : D^n \to D^n$ is a self-map of the closed n-disk. Construct S^n by identifying two copies D^n_+ and D^n_- of D^n using the identity map on the boundary S^{n-1}. Define a map $G : S^n \to S^n$ by $G(d) = g(d) \in D^n_+$ for $d \in D^n_+$ and also for $d \in D^n_-$. That is, both hemispheres are mapped to the hemisphere D^n_+. This map is continuous since the piecewise defintion agrees on the boundary S^{n-1}. Prove that G has a fixed point, and therefore that g has a fixed point. This is *Brouwer's fixed point theorem*

(5) Produce a CW complex as follows. Start with a solid cube, with the the polyhedral CW structure: there are eight 0-cells at the vertices, twelve 1-cells, being the edges, six 2-cells being the faces and a single interior 3-cell. Now form the quotient CW complex Q by identifying antipodal (closed) faces, but with a twist. For each pair of antipodal closed faces T and B, rotate the cube so T is on top and B on the bottom. Then identify T with B by twisting T by $\pi/2$ in the positive direction, and then projecting downwards. (Observe that the identification specified by this "anticlockwise twist" is the same whether T or B is placed on top.)

The resulting CW complex Q has one 3-cell, three 2-cells and some number of 1- and 0-cells. Determine the cell structure of Q and calculate $\mathrm{H}_*(Q;\mathbb{Z})$.