STABILIZED HP-DGFEM FOR INCOMPRESSIBLE FLOW

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We consider stabilized mixed $hp$-discontinuous Galerkin methods for the discretization of the Stokes problem in three-dimensional polyhedral domains. The methods are stabilized with a term penalizing the pressure jumps. For this approach it is shown that $Q_k - Q_{k-1}$ and $Q_k - Q_{k-1}$ elements satisfy a generalized inf-sup condition on geometric edge and boundary layer meshes that are refined anisotropically and non quasi-uniformly towards faces, edges, and corners. The discrete inf-sup constant is proven to be independent of the aspect ratios of the anisotropic elements and to decrease as $k^{-1/2}$ with the approximation order. We also show that the generalized inf-sup condition leads to a global stability result in a suitable energy norm.

Keywords: $hp$-FEM, discontinuous Galerkin methods, Stokes problem, anisotropic refinement

1. Introduction

Over the last few years, several discontinuous Galerkin (DG) methods for incompressible flow problems and for certain saddle-point problems with incompressibility constraints have been proposed in the literature. Here we only mention the piecewise solenoidal discontinuous Galerkin methods, the local discontinuous Galerkin (LDG) methods, and the interior penalty methods. The methods above all rely on discrete velocity spaces consisting of piecewise polynomial functions with no continuity constraints between the elements in the underlying triangulation. Inter elemental communication is achieved through so-called numerical fluxes, as in the original discontinuous Galerkin methods for non-linear hyperbolic systems. The main motivations for using DG methods in fluid flow problems lie in their robustness in convection-dominated regimes, their conservation properties, and their great flexibility in the mesh-design. Based on completely discontinuous finite element spaces, DG methods easily handle elements of various types and shapes, non-matching grids and even local spaces of different orders; they are therefore ideal for $hp$-adaptivity.

Even if transport phenomena may be dominant in incompressible flow problems, mixed DG methods still require suitable velocity-pressure pairs in order to ensure stability and convergence of the underlying Stokes discretization. It was shown recently that discontinuous $P_k - P_{k-1}$ and $Q_k - Q_{k-1}$ pairs are inf-sup stable with
respect to the mesh-size, as opposed to their conforming counterparts.\textsuperscript{18,29} These elements are optimal from an approximation point of view. A slightly different approach was proposed for the LDG methods.\textsuperscript{11,10} There, the introduction of a pressure stabilization term was proven to also render the convenient equal-order \( p - k \) and it was shown that several discontinuous velocity-pressure pairs possess better stability properties than their conforming versions.\textsuperscript{26} In particular, the numerical results reported in the experiments of Ref. 29 for two-dimensional uniform meshes show that discontinuous \( \mathbb{Q}_k - \mathbb{Q}_{k-1} \) elements are also uniformly stable with respect to the polynomial degree \( k \). For this pair, the best available bound of the inf-sup constant in terms of \( k \) was then shown to decrease as \( k^{-1} \), on shape-regular tensor-product meshes in two and three dimensions, possibly with hanging nodes.\textsuperscript{24} This bound ensures the same \( p \)-version convergence rate for the velocity and the pressure as that of conforming \( \mathbb{Q}_k - \mathbb{Q}_{k-2} \) elements in three dimensions. However, the latter elements are mismatched with respect to \( h \)-approximation.

In laminar regimes, solutions of incompressible flow problems in polyhedral domains have corner and edge singularities. In addition, strong boundary layers may arise at faces, edges, and corners. In the \( h p \)-version of the finite element method, these solution components can be approximated at exponential rate of convergence provided that the meshes are geometrically and anisotropically graded towards faces, edges, and corners.\textsuperscript{2,5,21,27,28} These anisotropically refined meshes raise serious stability issues in mixed approximations as the inf-sup constants might in general be very sensitive to the aspect ratios of the elements. It was recently shown for two- and three-dimensional conforming approximations employing \( \mathbb{Q}_k - \mathbb{Q}_{k-2} \) elements that, on corner, edge, and boundary-layer tensor-product meshes, the inf-sup constant for the Stokes problem is in fact independent of the aspect ratios of the anisotropic elements in the meshes.\textsuperscript{22,23,1,30} Recently, discontinuous \( \mathbb{Q}_k - \mathbb{Q}_{k-1} \) elements were studied on geometric edge meshes designed to resolve corner and edge singularities in the absence of boundary layers.\textsuperscript{25} By suitably defining the discontinuity stabilization parameters in the DG bilinear forms on anisotropic elements, it was proven that this velocity-pressure pair is divergence stable, with an inf-sup constant that is independent of the aspect ratios of the anisotropic elements and that decreases as \( k^{-3/2} \) in the approximation order.

In this paper, we analyze stabilized \( h p \)-discontinuous Galerkin methods on geometric meshes in three dimensions. We show that the introduction of the pressure stabilization term originally proposed for the LDG discretization\textsuperscript{11} leads to a generalized inf-sup constant for \( \mathbb{Q}_k - \mathbb{Q}_{k-1} \) and \( \mathbb{Q}_k - \mathbb{Q}_k \) elements that decreases only as \( k^{-1/2} \) in the polynomial degree, and is independent of possibly large aspect ratios of the mesh. As opposed to the work of Ref. 25 that only considers geometric edge meshes, the results here also hold for geometric boundary layer meshes that are additionally geometrically refined towards the faces. As a consequence of the generalized inf-sup condition, we obtain a global stability result in a suitable energy norm and derive \( p \)-version error bounds that are better than those available in the recent work of Ref. 24, by of half an order of \( k \) in the velocity and a full order of \( k \) in the pressure, respectively. We emphasize that, in our analysis, we use a similar unifying setting as that proposed in the analysis of Ref. 24. Thus, although we only consider the so-called interior penalty discontinuous Galerkin method, our results hold true verbatim for the analogues of the methods analyzed there, and, in particular, extend the LDG methods\textsuperscript{11,10} to the \( h p \)-context.

The outline of the paper is as follows. In Section 2, we introduce stabilized mixed \( h p \)-DGFEM for the Stokes problem. Two classes of geometric meshes are defined in Section 3. Continuity and coercivity properties of the DG forms on these meshes are established in Section 4. Our main result is the generalized inf-sup condition
that we present and prove in Section 5. A global stability result for the proposed DG discretizations is then derived in Section 6, together with \(hp\)-error bounds on shape-regular elements.

2. Stabilized mixed \(hp\)-DGFEM for the Stokes problem

In this section, we introduce stabilized mixed \(hp\)-discontinuous Galerkin methods using the pressure stabilization form that was originally proposed for the LDG discretization.\(^{11}\)

2.1. The Stokes problem

Let \(\Omega\) be a bounded polyhedron in \(\mathbb{R}^3\), and let \(\mathbf{n}\) be the outward normal unit vector to its boundary \(\partial \Omega\). Given a source term \(f \in L^2(\Omega)^3\) and a Dirichlet datum \(g \in H^{1/2}(\partial \Omega)^3\) with \(\int_{\partial \Omega} g \cdot \mathbf{n} \, ds = 0\), the Stokes problem consists in finding a velocity field \(\mathbf{u}\) and a pressure \(p\) such that

\[
\begin{align*}
-\nu \Delta \mathbf{u} + \nabla p &= f & \text{in } \Omega, \\
\nabla \cdot \mathbf{u} &= 0 & \text{in } \Omega, \\
\mathbf{u} &= g & \text{on } \partial \Omega.
\end{align*}
\]

(2.1)

Thanks to the continuous inf-sup condition\(^7\);\(^{16}\)

\[
\inf_{0 \neq q \in L^2(\Omega)/\mathbb{R}} \sup_{0 \neq v \in H^1_0(\Omega)^3} \frac{-\int_{\Omega} q \nabla \cdot v \, dx}{|v|_1 |q|_0} \geq C_\Omega > 0,
\]

(2.2)

with a constant \(C_\Omega\) depending only on \(\Omega\), the Stokes problem (2.1) has a unique solution \((\mathbf{u}, p)\) with

\[
\mathbf{u} \in \mathbf{V} := H^1(\Omega)^3, \quad p \in Q := L^2_0(\Omega) = L^2(\Omega)/\mathbb{R}.
\]

Here, we denote by \(|\cdot|_{s, D}\) and \(|\cdot|_{s, D}\) the norm and seminorm of the Sobolev space \(H^s(D), s \geq 0\), on a domain \(D\) in \(\mathbb{R}^d, d = 1, 2, 3\). The same notation is used to denote norms for vector fields. In case \(D = \Omega\), we drop the subscript.

2.2. Meshes and trace operators

Throughout, we consider triangulations \(\mathcal{T}\) on \(\Omega\) that consist of affine hexahedral elements \(\{K\}\). More precisely, each element \(K \in \mathcal{T}\) is obtained from the reference cube \(\hat{Q} = (-1, 1)^3\) by an affine mapping. In general, we allow for irregular meshes, i.e., meshes with hanging nodes (see, e.g., Sect. 4.4.1 of Ref. 26), but suppose that the intersection between neighboring elements is either a common vertex, a common edge, a common face, or an entire face of one of the two elements. An interior face of \(\mathcal{T}\) is the (non-empty) two-dimensional interior of \(\partial K^+ \cap \partial K^-\), where \(K^+\) and \(K^-\) are two adjacent elements of \(\mathcal{T}\). Similarly, a boundary face of \(\mathcal{T}\) is the (non-empty) two-dimensional interior of \(\partial K \cap \partial \Omega\) which consists of entire faces of \(\partial K\). We denote by \(\mathcal{F}_T\) the union of all interior faces of \(\mathcal{T}\), by \(\mathcal{F}_B\) the union of all boundary faces, and set \(\mathcal{F} = \mathcal{F}_T \cup \mathcal{F}_B\).

For an element \(K \in \mathcal{T}\), we denote its diameter by \(h_K\) and the radius of the largest circle that can be inscribed into \(K\) by \(\rho_K\). A mesh \(\mathcal{T}\) is called shape-regular if

\[
h_K \leq c \rho_K, \quad \forall K \in \mathcal{T},
\]

(2.3)

for a shape-regularity constant \(c > 0\) that is independent of the elements. As will be discussed below, our meshes are not necessarily shape-regular.
We next define some trace operators. Let $f \subset \mathcal{F}_T$ be an interior face shared by two elements $K^+$ and $K^-$ and $\mathbf{v}$, $\mathbf{q}$, and $\mathbf{r}$ be vector-, scalar- and matrix-valued functions, respectively, that are smooth inside each element $K^\pm$. We denote by $\mathbf{v}^\pm$, $\mathbf{q}^\pm$ and $\mathbf{r}^\pm$ the traces of $\mathbf{v}$, $\mathbf{q}$ and $\mathbf{r}$ on $f$ from the interior of $K^\pm$ and define the mean values $\langle \cdot \rangle$ and normal jumps $[\cdot]$ at $x \in f$ as

$$
\langle \mathbf{v} \rangle := (\mathbf{v}^+ + \mathbf{v}^-)/2, \quad \langle \mathbf{q} \rangle := (\mathbf{q}^+ + \mathbf{q}^-)/2, \quad \langle \mathbf{r} \rangle := (\mathbf{r}^+ + \mathbf{r}^-)/2,
$$

$$
\langle \mathbf{v} \rangle := \mathbf{v}^+ \otimes \mathbf{n}_{K^+} + \mathbf{v}^- \otimes \mathbf{n}_{K^-},
$$

where, for two vectors $\mathbf{a}$ and $\mathbf{b}$, $[\mathbf{a} \otimes \mathbf{b}]_{ij} = a_i b_j$. On a boundary face $f \subset \mathcal{F}_\Gamma$ given by $f = \partial K \cap \partial \Omega$, we set $\langle \mathbf{v} \rangle := \mathbf{v}$, $\langle \mathbf{q} \rangle := \mathbf{q}$, $\langle \mathbf{r} \rangle := \mathbf{r}$, as well as $[\mathbf{v}] := \mathbf{v} \cdot \mathbf{n}$, $\langle [\mathbf{v}] \rangle := \mathbf{v} \otimes \mathbf{n}$, $\langle \mathbf{q} \rangle := \mathbf{q} \cdot \mathbf{n}$ and $\langle [\mathbf{r}] \rangle := \mathbf{r} \cdot \mathbf{n}$.

### 2.3. Finite element spaces

Given a mesh $\mathcal{T}$ on $\Omega$ and an approximation order $k \geq 0$, we introduce the finite element spaces $V_h^k(\mathcal{T})$ and $Q_h^k(\mathcal{T})$:

$$
V_h^k(\mathcal{T}) := \{ \mathbf{v} \in L^2(\Omega)^3 : \mathbf{v}|_K \in Q_h(k)^3, \ K \in \mathcal{T} \},
$$

$$
Q_h^k(\mathcal{T}) := \{ q \in L^2(\Omega)^{D} : q|_K \in Q_h(k), \ K \in \mathcal{T} \},
$$

where $Q_h(k)$ is the space of polynomials of maximum degree $k$ in each variable on element $K$.

### 2.4. Mixed discontinuous Galerkin approximations

We approximate the velocities and pressures in the spaces $V_h$ and $Q_h$ given by

$$
V_h := V_h^k(\mathcal{T}), \quad Q_h := Q_h^k(\mathcal{T}),
$$

with $k \geq 1$ and $\ell = k$ or $\ell = k - 1$. We refer to these velocity-pressure pairs as (discontinuous) equal-order $Q_h-Q_h$ elements and mixed-order $Q_h-Q_{h-1}$ elements, respectively.

We consider the following stabilized mixed DG methods: find $(\mathbf{u}_h, p_h) \in V_h \times Q_h$ such that

$$
\begin{align*}
A_h(\mathbf{u}_h, \mathbf{v}) + B_h(\mathbf{v}, p_h) &= F_h(\mathbf{v}), \\
-B_h(\mathbf{u}_h, q) + C_h(p_h, q) &= G_h(q),
\end{align*}
$$

for all $(\mathbf{v}, q) \in V_h \times Q_h$. The forms $A_h$, $B_h$, and $C_h$ are given by

$$
A_h(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \nu \nabla_h \mathbf{u} : \nabla_h \mathbf{v} \, dx - \int_{\mathcal{E}} \langle \nu \nabla_h \mathbf{v} \rangle : [\mathbf{u}] + \langle \nu \nabla_h \mathbf{u} \rangle : [\mathbf{v}] \, ds \\
+ \nu \int_{\mathcal{E}} \delta \langle \mathbf{u} \rangle : [\mathbf{v}] \, ds,
$$

$$
B_h(\mathbf{v}, q) = - \int_{\Omega} q \nabla_h \cdot \mathbf{v} \, dx + \int_{\mathcal{E}} \langle q \rangle [\mathbf{v}] \, ds,
$$

$$
C_h(p, q) = \nu^{-1} \int_{\mathcal{E}} \gamma \langle p \rangle [q] \, ds.
$$
Here, $\nabla_h$ is the discrete gradient, taken elementwise. The functions $\delta \in L^\infty(F)$ and $\gamma \in L^\infty(F_T)$ are the so-called discontinuity and pressure stabilization functions, respectively, for which we will make a precise choice below. Finally, the corresponding right-hand sides $F_h$ and $G_h$ are

$$F_h(v) = \int_{\Omega} f \cdot v \, dx - \int_{F_S} (g \otimes n) \cdot \nabla_h v \, ds + \nu \int_{F_S} \delta g \cdot v \, ds,$$

$$G_h(q) = -\int_{F_S} q g \cdot n \, ds.$$

**Remark 2.1** It follows from the stability results below that problem (2.6) has a unique solution $(u_h, p_h) \in V_h \times Q_h$.

**Remark 2.2** The form $A_h(\cdot, \cdot)$ discretizing the Laplacian is the so-called interior penalty (IP) form. Several other choices are possible for $A_h(\cdot, \cdot)$. All the results of this paper also hold verbatim for the forms considered in the work of Ref. 24. The form $B_h(\cdot, \cdot)$ is related to the incompressibility constraint; it is used in several related mixed DG approaches. Finally, $C_h(\cdot, \cdot)$ is the pressure stabilization form that was originally introduced in the local discontinuous Galerkin methods.

### 2.5. Perturbed mixed formulation

For the purpose of the analysis, we introduce perturbed forms $\tilde{A}_h$ and $\tilde{B}_h$, following the ideas of Ref. 4 and of Ref. 24. To this end, we define the space $V(h) := V + V_h$, and introduce the lifting operators $\mathcal{L} : V(h) \to \Sigma_h$ and $\mathcal{M} : V(h) \to Q_h$ by

$$\int_{\Omega} \mathcal{L}(v) : \tau \, dx = \int_{\Omega} [v] : [\tau] \, ds, \quad \forall \tau \in \Sigma_h,$$

$$\int_{\Omega} \mathcal{M}(v) q \, dx = \int_{\Omega} [v] [q] \, ds, \quad \forall q \in Q_h,$$

where we use the auxiliary space $\Sigma_h = \{ \tau \in L^2(\Omega)^{3 \times 3} : \tau|_K \in Q_h(K)^{3 \times 3}, K \in T \}$. We then introduce the following perturbed forms on $V(h) \times V(h)$ and $V(h) \times Q$:

$$\tilde{A}_h(u, v) = \nu [\nabla_h u : \nabla_h v - \mathcal{L}(v) : \nabla_h v - \mathcal{L}(v) : \nabla_h v] \, dx + \nu \int_{F} \delta[u] : [v] \, ds,$$

$$\tilde{B}_h(v, q) = -\int_{\Omega} q [\nabla_h v - \mathcal{M}(v)] \, dx.$$

(2.7)

We have $\tilde{A}_h = A_h$ on $V_h \times V_h$, and $\tilde{B}_h = B_h$ on $V_h \times Q_h$, respectively. Thus, we may rewrite the method (2.6) as: find $(u_h, p_h) \in V_h \times Q_h$ such that

$$\begin{cases}
\tilde{A}_h(u_h, v) + \tilde{B}_h(v, p_h) = F_h(v) \\
-\tilde{B}_h(u_h, q) + C_h(p_h, q) = G_h(q)
\end{cases}$$

(2.8)

for all $(v, q) \in V_h \times Q_h$.

### 3. Geometric edge and boundary layer meshes

In this section, we define two classes of geometric meshes, namely geometric edges meshes that are employed in the presence of corner and edge singularities (as,
e.g., in Stokes flow or nearly incompressible elasticity), and geometric boundary layer meshes that are used when, in addition to corner/edge singularities, boundary layers are present as well. Both meshes are characterized by a geometric grading factor $\sigma \in (0, 1)$ and the number of layers $n$, the thinnest layer having width proportional to $\sigma^n$.

3.1. Geometric edge meshes

A geometric edge mesh $T_{\text{edge}}^{n,\sigma}$ is constructed by considering an initial shape-regular macro-triangulation $T_m = \{M\}$ of $\Omega$, with no hanging nodes, possibly consisting of just one element. The macro-elements $M$ in the interior of $\Omega$ are then refined isotropically and regularly (not discussed further) while the macro-elements $M$ on the boundary of $\Omega$ are refined geometrically and anisotropically towards edges and corners. This geometric refinement is obtained by affinely mapping corresponding reference triangulations (referred to as patches) on $\hat{Q}$ onto the macro-elements $M$ using the elemental maps $F_M : \hat{Q} \rightarrow M$. This process is illustrated in Figure 1. For edge meshes, the following patches on $\hat{Q} = I^3$, $I = (-1,1)$, are used for the geometric refinement towards the boundary of $\Omega$:

**Edge patches:** An edge patch $T_{\text{edge}}^{\text{edge}}$ on $\hat{Q}$ is given by

$$T_{\text{edge}}^{\text{edge}} := \{K_{xy} \times I \mid K_{xy} \in T_{xy}\},$$

where $T_{xy}$ is an irregular corner mesh, geometrically refined towards a vertex of $\hat{S} = (-1,1)^2$ with grading factor $\sigma$ and $n$ refinement levels; see Figure 1 (level 2, left).

**Corner patches:** In order to build a corner patch $T_{\text{edge}}^{\text{edge}}$ on $\hat{Q}$, we first consider an initial, irregular, corner mesh $T_{\text{corner}}$, geometrically refined towards a vertex of $\hat{Q}$, with grading factor $\sigma$ and $n$ refinement levels; see the mesh in bold lines in Figure 1 (level 2, right). The elements of this mesh are then irregularly refined towards the three edges adjacent to the vertex in order to obtain the mesh $T_{\text{edge}}^{\text{edge}}$; see also Figure 3.

For simplicity, we always assume that the only hanging nodes in geometric edge meshes $T_{\text{edge}}^{n,\sigma}$ are those in the closure of edge and corner patches.

![Figure 1: Hierarchic structure of a geometric edge mesh $T_{\text{edge}}^{n,\sigma}$. The macro-elements $M$ on the boundary of $\Omega$ (level 1) are further refined as edge and corner patches (level 2). The geometric grading factor is here $\sigma = 0.5$.](image-url)
3.2. **Geometric boundary layer meshes**

As for edge meshes, the construction of a geometric boundary layer mesh $T_{bl}^{n,\sigma}$ starts from an initial shape-regular macro-triangulation $T_m = \{ M \}$ of $\Omega$, with no hanging nodes, possibly consisting of just one element. The macro-elements $M$ on the boundary of $\Omega$ are now also refined geometrically towards faces; see Figure 2. More precisely, the following face, edge, and corner patches on $\tilde{Q} = I^3$ are used:

**Face patches:** A face patch $T_{bl}^{f}$ on $\tilde{Q}$ is given by an anisotropic triangulation of the form

$$T_{bl}^{f} := \{ K_x \times I \times I \ | \ K_x \in T_x \},$$

where $T_x$ is a mesh of $I$, geometrically refined towards one of the vertices, say $x = 1$, with grading factor $\sigma \in (0, 1)$ and total number of layers $n$; see Figure 2 (level 2, left).

**Edge patches:** An edge patch $T_{bl}^{e}$ on $\tilde{Q}$ is given by

$$T_{bl}^{e} := \{ K_{xy} \times I \ | \ K_{xy} \in \tilde{T}_{xy} \},$$

where $\tilde{T}_{xy}$ is a triangulation of $\tilde{S} = I^2$ obtained by first considering an irregular corner mesh $T_{xy}$ as in a patch $T_{c,xy}^{edge}$ of an edge mesh, geometrically refined towards a vertex of $\tilde{S}$, say $(x, y) = (1, 1)$, with grading factor $\sigma$ and $n$ refinement levels (see Figure 1 below, level 2, left). The elements of the mesh $T_{xy}$ are then anisotropically refined towards the two edges $x = 1$ and $y = 1$, in order to obtain a regular mesh $T_{xy}$. We refer to Figure 2 (level 2, center) for an example.

**Corner patches:** In order to build a corner patch $T_{bl}^{c}$ on $\tilde{Q}$, we first consider the same initial, irregular corner mesh $T_{c,m}$, geometrically refined towards a vertex of $\tilde{Q}$, with grading factor $\sigma$ and $n$ refinement levels; see the mesh in bold lines in Figure 2 (level 2, right). The elements of $T_{c,m}$ are then anisotropically refined towards the three faces $x = 1$, $y = 1$, and $z = 1$ in order to obtain a regular mesh $T_{bl}^{c}$; see also Figure 3.

For simplicity, we always assume that the three types of patches above are combined in such a way that geometric boundary layer meshes $T_{bl}^{n,\sigma}$ do not contain hanging nodes.

Figure 2: Hierarchic structure of a geometric boundary layer mesh $T_{bl}^{n,\sigma}$. The macro-elements $M$ on the boundary of $\Omega$ (level 1) are further refined as face, edge and corner patches (level 2). The geometric grading factor is here $\sigma = 0.5$. 

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Remark 3.1 We note that the underlying mesh $T_{c,m}$ is the same for the corner patches $T_{c}^{edge}$ and $T_{c}^{bl}$ in edge and boundary layer meshes, respectively. However, $T_{c}^{edge}$ is irregular and contains hanging nodes. Figure 3 shows the difference between corner patches for boundary layer and edge meshes.

![Figure 3: Geometrically refined corner patches $T_{c}^{bl}$ and $T_{c}^{edge}$ for boundary layer (left) and edge (right) meshes. The geometric grading factor is $\sigma = 0.5$.](image)

The geometric edge and boundary layer meshes defined above satisfy the following property; see Sect. 3 of Ref. 25.

**Property 3.2** Let $T$ be a geometric edge mesh $T_{edge}^{n,\sigma}$ or a geometric boundary layer mesh $T_{bl}^{n,\sigma}$, with a grading factor $\sigma \in (0,1)$ and $n$ levels of refinement. Then, any $K \in T$ can be written as $K = F_{K}(K_{xyz})$, where $K_{xyz}$ is of the form

$$K_{xyz} = I_{x} \times I_{y} \times I_{z} = (x_{1}, x_{2}) \times (y_{1}, y_{2}) \times (z_{1}, z_{2}),$$

and $F_{K}$ is an affine mapping, the Jacobian of which satisfies

$$|\det(J_{K})| \leq C, \quad |\det(J_{K}^{-1})| \leq C, \quad \|DF_{K}\| \leq C, \quad \|DF_{K}^{-1}\| \leq C,$$

with constants only depending on the angles of $K$ but not on its dimensions.

We note that the constants in Property 3.2 only depend on the shape-regularity constant in (2.3) of the underlying macro-element mesh $T_{m}$. The dimensions of $K_{xyz}$ on the other hand may depend on the geometric grading factor and the number of refinements.

For an element $K$ of a geometric edge mesh, we define, according to Property 3.2,

$$h_{x}^{K} = h_{x} = x_{2} - x_{1}, \quad h_{y}^{K} = h_{y} = y_{2} - y_{1}, \quad h_{z}^{K} = h_{z} = z_{2} - z_{1}.$$

### 3.3. Stabilization on geometric meshes

In this section, we define the discontinuity and pressure stabilization functions $\delta \in L^{\infty}(F)$ and $\gamma \in L^{\infty}(F_{T})$ on geometric meshes.

To this end, let $f$ be an entire face of an element $K$ of a geometric mesh $T$ on $\Omega$. According to Property 3.2, $K$ can be obtained by a stretched parallelepiped $K_{xyz}$ by an affine mapping $F_{K}$ that only changes the angles. Suppose that the face $f$ is the image of, e.g., the face $\{x = x_{1}\}$. We set $h_{f} = h_{x}$. For a face perpendicular to the $y$- or $z$-direction, we choose $h_{f} = h_{y}$ or $h_{f} = h_{z}$, respectively.
Let then $K$ and $K'$ be two elements with entire faces $f$ and $f'$ that share an interior face $f = f \cap f'$ in $F_I$. We have
\[ ch_f \leq h_{f'} \leq c^{-1} h_f, \]  
with a constant $c > 0$ that only depends on the geometric grading factor $\sigma$ and the constant in (2.3) for the underlying macro-mesh $T_m$. We define the function $h \in L^\infty(F)$ by
\[ h(x) := \begin{cases} \min\{h_f, h_{f'}\} & x \in f \cap f' \subset F_I, \\ h_f & x \in f \subset F_B. \end{cases} \]  
We then set
\[ \delta(x) = \delta_0 h^{-1} k^2, \quad x \in F, \]  
and
\[ \gamma(x) = \gamma_0 \min\{h_f, h_{f'}\} \max\{1, \ell\}^{-1}, \quad x \in F_I, \]  
with $\delta_0 > 0$ and $\gamma_0 > 0$ independent of $h$ and $k$.

**Remark 3.3** For isotropically refined and shape-regular meshes, the definitions in (3.2) and (3.3) are equivalent to the usual definition of $\delta$. Similarly, the definition of $\gamma$ in (3.4) generalizes the definition of Ref. 11 to the hp-version context on geometric meshes.

### 4. Continuity and coercivity on geometric meshes

On geometric meshes, the continuity of $A_h$ and $B_h$ as well as the coercivity of $A_h$ can be established as in Sect. 4 of Ref. 25.

To this end, we equip $V(h) = V + V_h$ with the broken norm
\[ \|v\|_h^2 := \sum_{K \in T} |v|^2_{1,K} + \int_F \delta|v|^2 ds, \quad v \in V(h). \]  
We have the following result.

**Theorem 4.1** Let $T$ be a geometric edge mesh $T_{edge}^{\sigma}$ or a geometric boundary layer mesh $T_{bl}^{\sigma}$, with a grading factor $\sigma \in (0, 1)$ and $n$ levels of refinement. Let the stabilization functions $\delta$ be defined as in (3.2) and (3.3). Then, the forms $A_h$ and $B_h$ in (2.7) are continuous:
\[ |A_h(v, w)| \leq \nu \alpha_1 \|v\|_h \|w\|_h \quad \forall v, w \in V(h), \]  
\[ |B_h(v, q)| \leq \alpha_2 \|v\|_h \|q\|_0 \quad \forall u \in V(h), \; q \in Q, \]  
with continuity constants $\alpha_1$ and $\alpha_2$ that depend on $\delta_0$ and the constants in Property 3.2, but are independent of $\nu$, $k$, $n$, and the aspect ratio of $T$. Furthermore, there exists a constant $\delta_{\text{min}} > 0$ that depends on the constants in Property 3.2, but is independent of $\nu$, $k$, $n$, and the aspect ratio of $T$, such that, for any $\delta_0 \geq \delta_{\text{min}},$
\[ A_h(v, v) \geq \nu \beta \|v\|_h^2 \quad \forall v \in V_h, \]  
for a coercivity constant $\beta > 0$ depending on $\delta_0$ and the constants in Property 3.2, but independent of $\nu$, $k$, $n$, and the aspect ratio of $T$. 

Remark 4.2 The results in Theorem 4.1 are based on anisotropic stability estimates for the lifting operators $L$ and $M$ that can be found in Sect. 4 of Ref. 25. These operators are identical for all the DG forms considered in the framework of Ref. 24 and, thus, the results in Theorem 4.1 also hold for all the mixed DG methods considered there. We note that the restriction on $\delta_0$ is typical for the interior penalty form $A_h$ and can be avoided if $A_h$ were chosen to be, e.g., the local discontinuous Galerkin form, the nonsymmetric interior penalty form, or the second Bassi-Rebay form.\textsuperscript{24}

Next, we address the continuity of $F_h$ and $G_h$.

**Theorem 4.3** Let $T$ be a geometric edge mesh $T_{\text{edge}}^{h,\sigma}$ or a geometric boundary layer mesh $T_{\text{bl}}^{h,\sigma}$, with a grading factor $\sigma \in (0,1)$ and $n$ levels of refinement. Let the stabilization functions be defined as in (3.2) and (3.3). Then we have

$$|F_h(v)| \leq C \left( \|f\|_0 + \nu \|\delta^{\frac{1}{2}} g\|_{0,\partial \Omega} \right) \|v\|_h \quad \forall v \in V_h,$$

$$|G_h(q)| \leq C \|\delta^{\frac{1}{2}} g\|_{0,\partial \Omega} \|q\|_0 \quad \forall q \in Q_h,$$

with continuity constants that depend on $\delta_0$, $\Omega$, and the constants in Property 3.2, but are independent of $\nu$, $k$, $n$, and the aspect ratio of $T$.

**Proof:** We first note that we have the Poincaré inequality

$$\|v\|_0 \leq C \|v\|_{1,h} \quad \forall v \in V(h), \quad (4.1)$$

with a constant depending on $\delta_0$, $\Omega$, and the constants in Property 3.2. The bound (4.1) follows by proceeding as in the original proof in Lemma 2.1 of Ref. 3, taking into account elliptic regularity theory for polyhedral domains and by using the anisotropic trace inequality

$$\|\varphi\|_{0,f} \leq C h_f^{-1} \|\varphi\|_{3/2+\varepsilon,K}, \quad \varepsilon > 0,$$

for an element $K \in T$ and an entire face $f$ of $\partial K$, with a constant depending on the constants in Property 3.2.

Let now $v \in V_h$. From (4.1), we obtain $\int_{\Omega} f \cdot v \, dx \leq C \|f\|_0 \|v\|_h$. Further, applying the discrete trace inequality from Lemma 3.3 of Ref. 25 as in the proof of Theorem 4.1 of Ref. 25,

$$\left| \int_{E_h} (g \otimes n) : (\nu \nabla_h v) \, ds \right| \leq C \nu \|\delta^{\frac{1}{2}} g\|_{0,\partial \Omega} \|v\|_h,$$

with a constant depending on $\delta_0$, and the constants in Property 3.2. Finally, the Cauchy-Schwarz inequality yields $\left| \int_{E_h} \delta g \cdot v \, ds \right| \leq \nu \|\delta^{\frac{1}{2}} g\|_{0,\partial \Omega} \|v\|_h$. This proves the assertion for $F_h$. Similarly, for $q \in Q_h$,

$$|G_h(q)| \leq \int_{E_h} q g \cdot n \, ds \leq \|\delta^{\frac{1}{2}} g\|_{0,\partial \Omega} \left( \int_{E_h} \delta^{-1} |q|^2 \, ds \right)^{\frac{1}{2}}.$$

Using again the techniques in Lemma 3.3 and Theorem 4.1 of Ref. 25, we have

$$\int_{E_h} \delta^{-1} |q|^2 \, ds \leq C \|q\|_{0,h}^2,$$

with a constant depending on $\delta_0$, and the constants in Property 3.2. This completes the proof. \hfill \Box
Remark 4.4 The same continuity properties hold for all the functionals $F_h$ and $G_h$ in the mixed DG methods analyzed in the setting of Ref. 24.

5. Generalized inf-sup condition on geometric meshes

Our main result establishes a generalized inf-sup condition on geometric meshes. To this end, we introduce the following seminorm on $Q_h$

$$|q|^2_{p_x} := \int_{p_x} \gamma |[q]|^2 ds,$$

with $\gamma$ the pressure stabilization function defined in (3.4).

We have the following result.

Theorem 5.5 Let $T$ be a geometric edge mesh $T_{edge}^{n,\sigma}$ or a geometric boundary layer mesh $T_{bl}^{n,\sigma}$, with a grading factor $\sigma \in (0, 1)$ and $n$ levels of refinement. Let the stabilization functions $\delta$ and $\gamma$ be defined according to (3.2), (3.3), and (3.4). Then, there exists a constant $C > 0$ that depends on $\Omega$, $\delta_0$, $\gamma_0$, and the constants in Property 3.2 and (3.1), but is independent of $h$, $k$, $\ell$, $n$, and the aspect ratio of $T$, such that, for any $n$ and $k \geq 1$, \( \ell = k \) or $\ell = k - 1$,

$$\sup_{0 \neq v \in \mathbf{v}_h} \frac{B_h(v, q)}{||v||_h} \geq C k^{-\frac{1}{2}} ||q||_0 (1 - |[q]|_{p_x})^2, \quad \forall q \in Q_h \setminus \{0\}.$$

Remark 5.6 For $h$-version DG approximations on shape-regular meshes, the generalized inf-sup condition in Theorem 5.5 was established recently for the LDG discretizations, in a form that also involves the auxiliary stresses present in the LDG approach. Similar inf-sup conditions also play an important role in the analysis of conforming stabilized mixed methods.

The proof of Theorem 5.5 is carried out in the rest of this section. We begin by collecting several properties of $L^2$-projections and by deriving bounds for averages and jumps over faces of geometric meshes. We then complete the proof of Theorem 5.5.

5.1. $L^2$-projections

For an interval $I_x = (x_1, x_2)$, let $\Pi_x : L^2(I_x) \to Q_k(I_x)$ denote the one-dimensional $L^2$-projection onto the space $Q_k(I_x)$ of polynomials of degree at most $k$ on $I_x$; given $v \in L^2(I_x)$, this projection is defined by imposing

$$\int_{I_x} \Pi_x v \varphi dx = \int_{I_x} v \varphi dx, \quad \forall \varphi \in Q_k(I_x).$$

The $L^2$-projection is stable:

$$||\Pi_x v||_{0, I_x} \leq ||v||_{0, I_x}, \quad \forall v \in L^2(I_x). \quad (5.2)$$

Moreover, applying similar techniques as in Theorem 2.2 of Ref. 8, we have, for $k \geq 1$,

$$||\Pi_x v||_{1, I_x} \leq C k^{-\frac{1}{2}} ||v||_{1, I_x}, \quad \forall v \in H^1(I_x), \quad (5.3)$$

with a constant $C > 0$ independent of $k$, $I_x$, and $v$.

We next recall the following approximation result from Lemma 3.5 of Ref. 19.
Lemma 5.7 \(\text{Let} \ I_x = (x_1, x_2), \ h_x = x_2 - x_1 \text{ and } v \in H^1(I_x). \text{ Then, there holds} \)
\[
|v(x_1) - \Pi_x v(x_1)|^2 + |v(x_1) - \Pi_x v(x_2)|^2 \leq C h_x k^{-1} \|v\|^2_{0,I_x},
\]
\(\text{for } k \geq 1 \text{ and with a constant } C > 0 \text{ independent of } h_x, k, \text{ and } v.\)

We will also make use of an approximation result from Lemma 3.9 of Ref. 19 for the two-dimensional \(L^2\)-projection \(\Pi_x \otimes \Pi_y; \) here, the subscripts indicate the variables the projections \(\Pi_x \) and \(\Pi_y \) act on.

**Lemma 5.8** \(\text{Let} \ I_x = (x_1, x_2), \ I_y = (y_1, y_2), \ h_x = x_2 - x_1 \text{ and } h_y = y_2 - y_1. \text{ Assume that there exists a constant } c > 0 \text{ such that } c h_x \leq h_y \leq c^{-1} h_x. \text{ Then, for} \)
\(v \in H^1(I_x \times I_y) \text{ and } k \geq 1, \) we have
\[
\|v - \Pi_x \otimes \Pi_y v\|^2_{0,\partial(I_x \times I_y)} \leq C h_x k^{-1} \|v\|^2_{I_x \times I_y},
\]
\(\text{with a constant } C > 0 \text{ depending on } c, \text{ but independent of } h_x, h_y, k, \text{ and } v. \)

For an axiparallel element \(K_{xyz} = (x_1, x_2) \times (y_1, y_2) \times (z_1, z_2), \) the \(L^2\)-projection \(\Pi_{K_{xyz}} : L^2(K_{xyz}) \rightarrow \mathbb{Q}_k(K_{xyz}) \) is the product operator \(\Pi_{K_{xyz}} = \Pi_x \otimes \Pi_y \otimes \Pi_z \) of one-dimensional \(L^2\)-projections. For \(v \in L^2(K_{xyz}), \) it satisfies
\[
\int_{K_{xyz}} \Pi_{K_{xyz}} v \phi \, dx = \int_{K_{xyz}} v \phi \, dx, \quad \forall \phi \in \mathbb{Q}_k(K_{xyz}).
\]

For an element \(K\) of a geometric edge or boundary layer mesh \(T, \) the \(L^2\)-projection \(\Pi_K : L^2(K) \rightarrow \mathbb{Q}_k(K) \) is defined by
\[
\int_K \Pi_K v \phi \, dx = \int_K v \phi \, dx, \quad \forall \phi \in \mathbb{Q}_k(K).
\]

Thanks to Property 3.2, we have \(K = F_K(K_{xyz}) \) for an axiparallel element \(K_{xyz} = (x_1, x_2) \times (y_1, y_2) \times (z_1, z_2). \) For \(v \in L^2(K), \) we therefore have
\[
\Pi_K v \circ F_K = \Pi_{K_{xyz}} [v \circ F_K], \quad \text{on } K_{xyz}. \quad (5.4)
\]

We have the following stability result.

**Lemma 5.9** \(\text{Let } T \text{ be a geometric edge or boundary layer mesh. Let } K \in T \text{ and } v \in H^1(K). \text{ Then we have for } k \geq 1 \)
\[
\|\Pi_K v\|_{1,K} \leq C k^{\frac{3}{2}} |v|_{1,K},
\]
\(\text{with a constant } C > 0 \text{ depending on the bounds in Property 3.2, but independent of} \ k, \ K, \text{ and } v. \)

**Proof:** \(\text{Let } K = K_{xyz} \text{ according to Property 3.2. The bounds (5.2) and (5.3) imply that} \)
\[
\|\partial_z \Pi_{K_{xyz}} v\|_{0,K_{xyz}} \leq C k^{\frac{3}{2}} \|\partial_z v\|_{0,K_{xyz}}, \]
\[
\|\partial_y \Pi_{K_{xyz}} v\|_{0,K_{xyz}} \leq C k^{\frac{3}{2}} \|\partial_y v\|_{0,K_{xyz}}, \]
\[
\|\partial_x \Pi_{K_{xyz}} v\|_{0,K_{xyz}} \leq C k^{\frac{3}{2}} \|\partial_x v\|_{0,K_{xyz}},
\]
\(\text{for any } v \in H^1(K_{xyz}), \text{ with a constant } C > 0 \text{ independent of } k, K_{xyz}, \text{ and } v. \) A scaling argument and the bounds in Property 3.2 prove the assertion for a general element \(K. \quad \square\)
Finally, the $L^2$-projection $\Pi : L^2(\Omega) \to \{v \in L^2(\Omega) : v|_K \in Q_h(K), \ K \in T\}$ is defined elementwise by $\Pi v|_K = \Pi_K v|_K, \ K \in T$. For vector fields, we use bold-face notation (such as $\Pi_{K_{xyz}}, \Pi_K$, and $\Pi$) to denote the $L^2$-projections that are applied componentwise.

5.2. Auxiliary results

In this section, we derive bounds for the averages and jumps over faces. We start by considering interior faces.

5.2.1. Interior faces

Let $K$ and $K'$ be two elements of a geometric mesh with entire faces $f$ and $f'$ that share an interior face $f \cap f'$. We may assume that $f \cap f'$ is an entire face of $K$, that is, $f \cap f' = f$. By Property 3.2, we have $K = F_K(K_{xyz})$ and $K' = F_{K'}(K'_{xyz})$ with, e.g.,

$$K_{xyz} = (x_1, x_2) \times (y_1, y_2) \times (z_1, z_2), \quad K'_{xyz} = (x_2, x_3) \times (y_1, y_3) \times (z_1, z_2),$$

and $y_2 \leq y_3$. The face $f$ is then given by $f = F_f(f_{xyz})$, with $f_{xyz} = \{x_2\} \times \{y_1, y_2\} \times (z_1, z_2)$, and $F_f(y, z) = F_K(x_2, y, z) = F_{K'}(x_2, y, z)$ for $y_1 \leq y \leq y_2, z_1 \leq z \leq z_2$. Similarly, we have $f' = F_{f'}(f'_{xyz})$. For a function $v \in H^1(K \cup K')^3$, we define $v_{xyz} = v|_K \circ F_K$ and $v'_{xyz} = v|_{K'} \circ F_{K'}$.

We set $h_x = x_2 - x_1$, $h'_x = x_3 - x_2$, $h_y = y_2 - y_1$, $h'_y = y_3 - y_1$, and $h_z = z_2 - z_1$, and may assume that

$$c h_x \leq h'_x \leq c^{-1} h_x, \quad (5.5)$$

according to (3.1). In the case where the elements $K$ and $K'$ match regularly (i.e., $f = f'$) the ratios of the mesh-sizes $h_x, h_y$ and $h_z$ can be arbitrary. However, when $K$ and $K'$ match irregularly (i.e., $f \neq f'$), it is essential to observe that, by definition of geometric meshes, we also have

$$c h_y \leq h'_y \leq c^{-1} h_y, \quad c h_z \leq h'_z \leq c^{-1} h_z, \quad (5.6)$$

with a constant $c > 0$ depending solely on the bounds in Property 3.2. The situation when $K$ and $K'$ match irregularly is shown in Figure 4. We point out that the above configuration covers all interior faces in geometric edge and boundary layer meshes.

We first show the following result.

**Lemma 5.10** Let $K, K' \in T$ share a face $f \subset F_T$. Then, for $q \in Q_h$ and $v \in H^1(K \cup K')^3$,

$$\left| \int_f [q] \cdot [v - \Pi v] \, ds \right| \leq C \left( \int_f \gamma \|q\|_1^2 \, ds \right)^{\frac{1}{2}} \left( \|v\|_{1,K}^2 + \|v\|_{1,K'}^2 \right)^{\frac{1}{2}},$$

with a constant $C > 0$ that depends on $\gamma_0$ and the bounds in Property 3.2 and (3.1).

**Proof:** We begin by noting that $\int_f [q] \cdot [v - \Pi v] \, ds = \frac{1}{2} S_K + \frac{1}{2} S_{K'}$, where

$$S_K = \int_f [q] \cdot (v|_K - \Pi_K v|_K) \, ds,$$

$$S_{K'} = \int_f [q] \cdot (v|_{K'} - \Pi_{K'} v|_{K'}) \, ds.$$
Step 1: We start by bounding the term $S_K$. Setting $q_{yz} = [q] \circ F_f$, we obtain

\[
S_K = \int f_{yz} [q] \cdot (v|_K - \Pi_K v|_K) \, ds
\]

\[
= \int f_{yz} q_{yz} \cdot (v_{xy} - \Pi_{xy} v_{xy}) \, dy \, dz
\]

\[
= \int f_{yz} q_{yz} \cdot (v_{xy} - \Pi_x \otimes \Pi_y \otimes \Pi_z v_{xy}) \, dy \, dz
\]

Here, we have used identity (5.4), the factorization $\Pi_{xy} = \Pi_x \otimes \Pi_y \otimes \Pi_z$ into one-dimensional $L^2$-projections, and the fact that each component of $q_{yz}$ is a polynomial of degree $\ell = k$ or $\ell = k - 1$ in $y$- and $z$-direction. The Cauchy-Schwarz inequality, the definition of $h_f$ in (3.1), the definition of $\gamma$ in (3.4), and (5.5) yield

\[
S_K \leq T_K \left( \int f_{yz} h_x \max\{1, \ell\}^{-1} |q_{yz}|^2 \, |\det(DF_f)| \, dy \, dz \right)^{\frac{1}{2}}
\]

\[
\leq C \cdot T_K \cdot \left( \int f |[q]|^2 \, ds \right)^{\frac{1}{2}},
\]

with the term $T_K$ given by

\[
T_K := \max\{1, \ell\} h_x^{-1} \int f_{yz} |\Pi_y \otimes \Pi_z v_{xy} - \Pi_x \otimes \Pi_y \otimes \Pi_z v_{xy}|^2 \, |\det(DF_f)| \, dy \, dz.
\]

From the stability of the one-dimensional projections $\Pi_y$ and $\Pi_z$ in (5.2) (taking into account that $|\det(DF_f)|$ is constant), the approximation result in Lemma 5.7.
and the bounds in Property 3.2, we obtain
\[
T_{K'}^2 \leq \max\{1, \ell\} h_x^{-1} |\det(DF_f)| \int_{f_{yz}} |v_{xyz} - \Pi_x v_{xyz}|^2 dy dz \\
\leq C \max\{1, \ell\} h_x^{-1} |\det(DF_f)| \|\partial_x v_{xyz}\|_{0,K_{xyz}}^2 \\
\leq C |\det(DF_f)| \|DF_K\| \|\det(DF_K^{-1})\| \|v_{1,K'}^2 \leq C |v|^2_{1,K}.
\]
Combining the above estimates shows that
\[
S_K \leq C |v|_{1,K} \left( \int_f \gamma |\|q\| |^2 ds \right)^{1/2}, \tag{5.7}
\]
with a constant $C$ depending on $\gamma_0$ and the bounds in Property 3.2 and (3.1).

**Step 2:** Let us now consider the term $S_{K'}$. We note that there is an entire face $f'$ of $K'$, such that $f = f \cap f'$. If $f = f'$, then $S_{K'}$ can be bounded as $S_K$ in Step 1. Thus, we only need to consider the case where $f$ is an irregular face of $K'$, i.e., $f$ is a proper subset of $f'$ as in Figure 4. As in the proof of Step 1, since $q_{yz}$ is a polynomial in z-direction, we have:
\[
S_{K'} = \int_f [\gamma] \cdot (v|_{K'} - \Pi_{K'} v|_{K'}) ds \\
= \int_{f_{yz}} [q_{yz}] \cdot (v'_{xyz} - \Pi'_{xyz} v'_{xyz}) |\det(DF_f)| dy dz \\
= \int_{f_{yz}} [q_{yz}] \cdot (\Pi'_{x} v'_{xyz} \Pi'_{y} v'_{xyz} - \Pi'_{x} \Pi'_{y} v'_{xyz}) |\det(DF_f)| dy dz.
\]
Here, we denote by $\Pi'_{x}, \Pi'_{y},$ and $\Pi'_{z}$ the one-dimensional $L^2$-projections on $K'_{xyz}$. We obtain
\[
S_{K'} \leq C \cdot T_{K'} \cdot \left( \int_f \gamma |\|q\| |^2 ds \right)^{1/2},
\]
with the term $T_{K'}$ given by
\[
T_{K'}^2 := \max\{1, \ell\} h_x^{-1} \int_{f_{yz}} |\Pi'_{x} v'_{xyz} - \Pi'_{x} \Pi'_{y} v'_{xyz}|^2 |\det(DF_f)| dy dz.
\]
From the stability (5.2) of $\Pi'_{z}$ in z-direction, (5.5), (5.6) and Lemma 5.8, we obtain
\[
T_{K'}^2 \leq \max\{1, \ell\} h_x^{-1} |\det(DF_f)| \int_{f_{yz}} |v'_{xyz} - \Pi'_{x} \Pi'_{y} v'_{xyz}|^2 dy dz \\
\leq C |v'_{xyz}|_{1,K_{xyz}}^2 \leq |v|^2_{1,K'}.
\]
Combining the bounds above gives
\[
S_{K'} \leq C |v|_{1,K'} \left( \int_f \gamma |\|q\| |^2 ds \right)^{1/2}, \tag{5.8}
\]
with a constant $C > 0$ depending on $\gamma_0$ and the bounds in Property 3.2 and (3.1). Combining (5.7) and (5.8) concludes the proof.

Next, we estimate the jump of the $L^2$-projection over the face $f$. 

\textbf{Lemma 5.11} Let $K, K' \in T$ share a face $f \subset F_T$. Suppose that $f = f \cap f'$, with $f$ and $f'$ entire faces of $K$ and $K'$, respectively. Then, for $v \in H^1(K \cup K')^3$, 

$$
\int_f \|\Pi v\|^2 ds \leq C \min\{h_f, h_{f'}\}^{-1} \left[|v|^2_{1, K} + |v|^2_{1, K'}\right],
$$

with a constant $C > 0$ that depends only on the bounds in Property 3.2 and (3.1).

\textbf{Proof:} Equality (5.4) ensures that 

$$
\int_f \|\Pi v\|^2 ds \leq \int_f |\Pi_K v|_{\|\cdot\|_{1,K}}^2 ds = \int_{f_{yz}} |\Pi_{K_{xyz}} v_{xyz} - \Pi_{K'_{xyz}} v'_{xyz}|^2 |\text{det}(DF_f)| dy dz.
$$

We consider two cases separately.

\textbf{Case 1:} Let $K$ and $K'$ match regularly, i.e., $f = f'$. Since $v_{xyz}$ and $v'_{xyz}$ coincide on the face $f_{yz}$, we have $\Pi_y \otimes \Pi_z v_{xyz} = \Pi'_y \otimes \Pi'_z v'_{xyz}$ on $f_{yz}$. We thus obtain from the triangle inequality 

$$
\int_f \|\Pi v\|^2 ds \leq C \cdot |\text{det}(DF_f)| \cdot [T_K + T_{K'}],
$$

with 

$$
T_K = \int_{f_{yz}} |\Pi_y \otimes \Pi_z v_{xyz} - \Pi_z \otimes \Pi_y \otimes \Pi_z v_{xyz}|^2 dy dz,
$$

$$
T_{K'} = \int_{f_{yz}} |\Pi'_y \otimes \Pi'_z v'_{xyz} - \Pi'_z \otimes \Pi'_y \otimes \Pi'_z v'_{xyz}|^2 dy dz.
$$

Using the stability (5.2) of the projections $\Pi_y$ and $\Pi_z$ in $y$- and $z$-directions, as well as the approximation result in Lemma 5.7, we obtain 

$$
T_K \leq Ch_k^{-1} |\partial_z v_{xyz}|^2_{0, K_{xyz}} \leq Ch_k^{-1} |v|^2_{1, K}.
$$

An analogous bound for $T_{K'}$ and (5.5) prove the assertion in this case.

\textbf{Case 2:} Assume that $K$ and $K'$ are non-matching ($f \neq f'$). We then have that $\Pi_z v_{xyz} = \Pi'_z v'_{xyz}$ on $f_{yz}$. Thus, 

$$
\int_f \|\Pi v\|^2 ds \leq C \cdot |\text{det}(DF_f)| \cdot [T_K + T_{K'}],
$$

with 

$$
T_K = \int_{f_{yz}} |\Pi_z v_{xyz} - \Pi_z v_{xyz}|^2 dy dz,
$$

$$
T_{K'} = \int_{f_{yz}} |\Pi'_z v'_{xyz} - \Pi'_z v'_{xyz}|^2 dy dz.$$


Since the underlying elements are shape-regular in $x$- and $y$-directions thanks to (5.5) and (5.6), we can invoke the stability (5.2) of $\Pi_z$ and the approximation result in Lemma 5.8. This gives

$$T_{K'} \leq \int_{f_{yz}} |v'_{xyz} - \Pi'_x \otimes \Pi'_y v'_{xyz}|^2 dy dz$$

$$\leq \int_{f_{yz}} |v'_{xyz} - \Pi'_x \otimes \Pi'_y v'_{xyz}|^2 dy dz$$

$$\leq Ch'_xf^{-1} |v'_{xyz}|_{1, K_{xyz}}^2 \leq Ch'_xf^{-1} |v|_{1, K'}^2.$$ 

An analogous bound for $T_K$ and (5.5) prove the assertion in this case. 

5.2.2. Boundary faces

We conclude by stating an analogous result to Lemma 5.11 for boundary faces that can be proved with exactly the same techniques. Let $K$ be an element on the boundary and $f$ an entire face of $K$ in $\mathcal{F}_B$.

**Lemma 5.12** For $v \in H^1_0(K)^3$, we have

$$\int_f [\Pi v]^2 ds \leq C h_f k^{-1} |v|_{1, K}^2,$$

with a constant $C > 0$ depending on the bounds in Property 3.2.

5.3. Proof of Theorem 5.5

Fix $q \in Q_h$. From the continuous inf-sup condition (2.2), there exists a field $w \in H^1_0(\Omega)^3$ such that

$$-\int_\Omega q \nabla \cdot w \, dx = \|q\|_0^2, \quad |w|_1 \leq C_{\Omega}^{-1} \|q\|_0,$$  \hspace{1cm} (5.9)

where $C_{\Omega} > 0$ is the continuous inf-sup constant. We then set $v = \Pi w$, with $\Pi$ the $L^2$-projection defined previously. Using $[w] = 0$ on $\mathcal{F}$, (5.9), integration by parts, and the properties of the $L^2$-projection, we find

$$B_h(v, q) = B_h(w, q) + B_h(\Pi w - w, q)$$

$$= \|q\|_0^2 + \int_\Omega \nabla q \cdot (\Pi w - w) \, dx - \int_{\mathcal{F}_I} [q] \cdot \{\Pi w - w\} \, ds$$

$$= \|q\|_0^2 + \int_{\mathcal{F}_I} [q] \cdot \{w - \Pi w\} \, ds.$$ 

Applying Lemma 5.10 gives

$$\left| \int_{\mathcal{F}_I} [q] \cdot \{w - \Pi w\} \, ds \right| \leq \sum_{f \in \mathcal{F}_I} \left| \int_f [q] \cdot \{w - \Pi w\} \, ds \right|$$

$$\leq C \left( \sum_{K \in T} |w|_{1, K}^2 \right)^{1/2} \left( \sum_{f \in \mathcal{F}_I} \gamma_f |[q]|^2 \, ds \right)^{1/2}$$

$$\leq C |w|_1 |q|_{\mathcal{F}_I}.$$ 

Combining the above estimates with (5.9) yields

\[ B_h(v, q) \geq C \|q\|_0^2 \left( 1 - \frac{|q|_{F_T}}{\|q\|_0} \right), \]  

(5.10)

with a constant \( C > 0 \) depending on \( C_\Omega, \gamma_0 \), and the bounds in Property 3.2 and (3.1).

We have from Lemma 5.9, Lemma 5.11 and Lemma 5.12, together with the definition of the discontinuity stabilization function \( \delta \),

\[ \|v\|_h^2 = \sum_{K \in T} |\Pi w|_{1, K}^2 + \sum_{f \in \mathcal{F}} \int_f \delta |\Pi w|_F^2 \, ds \]

\[ \leq Ck \sum_{K \in T} |w|_{1, K}^2 + Ck \sum_{K \in T} |w|_{1, K}^2 \leq Ck |w|_1^2. \]

Thus, invoking (5.9),

\[ \|v\|_h \leq C k^{\frac{1}{2}} \|q\|_0. \]  

(5.11)

Combining (5.10) and (5.11) concludes the proof of Theorem 5.5.

6. Global stability and a-priori error estimates

In this section, we show how the stability results in the previous sections can be used to obtain a global stability result and to derive a-priori error estimates. The technique we use is closely related to that used in the analysis of conforming stabilized mixed methods.\textsuperscript{15,14}

6.1. Global stability

Let \( W_h \) be the product space \( W_h = V_h \times Q_h \), endowed with the norm

\[ \|(v, q)\|_{DG}^2 = \nu \|v\|_h^2 + \nu^{-1} k^{-1} \|q\|_0^2 + \nu^{-1} |q|_{F_T}^2. \]

In \( W_h \) we define the forms

\[ A_h(u, p; v, q) = \tilde{A}_h(u, v) + \tilde{B}_h(v, p) - B_h(u, q) + C_h(p, q), \]

\[ L_h(v, q) = F_h(v) + G_h(q), \]

and reformulate (2.8) equivalently as: find \((u_h, p_h) \in W_h\) such that

\[ A_h(u_h, p_h; v, q) = L_h(v, q) \]  

(6.1)

for all \((v, q) \in W_h\).

The following stability result holds.

**Theorem 6.1** Let \( T \) be a geometric edge mesh \( T^{n, \sigma}_{edge} \) or a geometric boundary layer mesh \( T^{n, \sigma}_b \), with a grading factor \( \sigma \in (0, 1) \) and \( n \) levels of refinement. Let the stabilization functions \( \delta \) and \( \gamma \) be defined according to (3.2), (3.3), and (3.4). Then, there exists a constant \( C > 0 \) that depends on \( \Omega, \delta_0, \gamma_0 \), and the constants in Property 3.2 and (3.1), but is independent of \( \nu, k, \ell, n \), and the aspect ratio of of \( T \), such that, for any \( n \) and \( k \geq 1, \ell = k \) or \( \ell = k - 1, \)

\[ \inf_{(0, 0) \neq (u, p) \in W_h} \sup_{(v, q) \in W_h} \frac{A_h(u, p; v, q)}{\|(u, p)\|_{DG} \|(v, q)\|_{DG}} \geq C. \]
Proof: Fix \((0,0) \neq (u,p) \in V_h \times Q_h\). Thanks to the coercivity of \(A_h\) in Theorem 4.1 and the definition of \(C_h\), we have
\[
A_h(u, p; u, p) \geq \nu \beta \|u\|^2_h + \nu^{-1} \|p\|^2_{\mathcal{F}_h}. \tag{6.2}
\]
Furthermore, Theorem 5.5 guarantees the existence of a velocity \(w \in V_h\) satisfying
\[
B_h(w, p) \geq C \|p\|^2_0 - C \|p\|_{\mathcal{F}_h} \|p\|_0, \quad \|w\|_h \leq C k^{\frac{1}{2}} \|p\|_0. \tag{6.3}
\]
From the definition of \(A_h\), the continuity properties in Theorem 4.1, weighted Cauchy-Schwarz inequalities, and (6.3), we obtain
\[
A_h(u, p; w, 0) = A_h(u, w) + B_h(w, p) \\
\geq -C \epsilon_1 \nu \|u\|^2_h - C \epsilon_2 \|w\|^2_h + C \|p\|^2_0 - C \epsilon_2 \|p\|_{\mathcal{F}_h}^2 \\
\geq C(1 - \epsilon_2^{-1} - \nu \epsilon_1^{-1}) \|p\|^2_0 - C \epsilon_1 \nu \|u\|^2_h - C \epsilon_2 \|p\|_{\mathcal{F}_h}^2, \tag{6.4}
\]
with parameters \(\epsilon_1, \epsilon_2 > 0\) at our disposal. We next set \((v, q) = (u, p) + \epsilon_3(w, 0)\), with \(\epsilon_3 > 0\). Then, combining (6.2) and (6.4), yields
\[
A_h(u, p; v, q) \geq C \nu(1 - \epsilon_3 \epsilon_2) \|u\|^2_h + C \nu \epsilon_3^2 \|w\|^2_h + C \nu \epsilon_3 \|p\|^2_{\mathcal{F}_h} + C \nu \epsilon_3 \|p\|_{\mathcal{F}_h}^2 \tag{6.8}
\]
It is now easy to see that one can select \(\epsilon_1\) of order \(O(ku)\), \(\epsilon_2\) of order \(O(k)\), and \(\epsilon_3\) of order \(O(\nu^{-1}k^{-1})\), respectively, in such a way that
\[
A_h(u, p; v, q) \geq C \nu \|u\|^2_h + C \nu \|p\|^2_{\mathcal{F}_h} + C \nu \|p\|_{\mathcal{F}_h}^2 = C \|(u, p)\|^2_{DG}. \tag{6.5}
\]
Using the fact that \(\epsilon_3\) is of order \(O(\nu^{-1}k^{-1})\) and (6.3) give
\[
\|(v, q)\|^2_{DG} \leq C \nu \|u\|^2_h + C \nu \|w\|^2_h + C \nu \|p\|_{\mathcal{F}_h}^2 + \nu^{-1} \|p\|_{\mathcal{F}_h}^2 \\
\leq C \nu \|u\|^2_h + C \nu \|w\|^2_h + \nu^{-1} \|p\|_{\mathcal{F}_h}^2 \\
\leq C \|(u, p)\|^2_{DG}. \tag{6.6}
\]
Combining (6.5) and (6.6) completes the proof. \(\square\)

6.2. A-priori error estimates

In order to derive a-priori error estimates, we let \((u, p)\) be the exact solution of the Stokes system (2.1) and assume that \(p \in H^1(\Omega_{int})\) in a domain \(\Omega_{int} \subset \Omega\) containing all the interior faces in \(\mathcal{F}_h\). Thus, \([p] = 0\) on \(\mathcal{F}_h\). We define \(Q(h) := Q_h + H^1(\Omega_{int})\) and \(W(h) := V(h) \times Q(h)\), equipped with the norm \(\|(v, q)\|_{DG}\).

From the continuity properties in Theorem 4.1, Theorem 4.3 and the Cauchy-Schwarz inequality, it can be seen that
\[
|A_h(u, p; v, q)| \leq C k^{\frac{1}{2}} \|(u, p)\|_{DG} \|(v, q)\|_{DG}, \quad \forall (u, p), (v, q) \in W(h), \tag{6.7}
\]
and
\[
|\mathcal{L}(v, q)| \leq C \left[\nu^{-1} \|f\|^2_0 + \nu k \|\delta^{-1}g\|_{0,0h}^2 \right]^{\frac{1}{2}} \|(v, q)\|_{DG}, \quad \forall (v, q) \in W(h), \tag{6.8}
\]
with constants as in Theorem 4.1 and Theorem 4.3, respectively.

Taking into account (6.7), the global inf-sup condition in Theorem 6.1, and the non-consistency of the forms \(A_h\) and \(B_h\), we obtain straightforwardly the following a-priori bound.
Corollary 6.2 Let \((\mathbf{u}, p)\) be the exact solution of the Stokes system (2.1), with \(p \in H^1(\Omega_{\text{int}})\), and let \((\mathbf{u}_h, p_h)\) be its discontinuous Galerkin approximation (2.6) on a geometric edge mesh \(T = T^{n,\sigma}_{\text{edge}}\) or a geometric boundary layer mesh \(T = T^{n,\sigma}_{\text{bl}}\), with a grading factor \(\sigma \in (0,1)\) and \(n\) levels of refinement. Let the stabilization functions \(\delta\) and \(\gamma\) be defined as in (3.2), (3.3) and (3.4), respectively. Then,

\[
\|\mathbf{u} - \mathbf{u}_h, p - p_h\|_{DG} \leq C h^\frac{1}{2} \inf_{(v, q) \in \mathbf{W}_h} \|\mathbf{u} - v, p - q\|_{DG} + C R_h(\mathbf{u}, p),
\]

with a constant \(C > 0\) that depends on \(\Omega, \delta_0, \gamma_0\), and the constants in Property 3.2 and (3.1), but is independent of \(v, k, \ell, n\), and the aspect ratio of the anisotropic elements in \(T\). Here, \(R_h(\mathbf{u}, p)\) is the residual

\[
R_h(\mathbf{u}, p) = \sup_{(w, s) \in \mathbf{W}_h} \frac{|\mathcal{A}_h(\mathbf{u}, p; w, s) - \mathcal{L}_h(w, s)|}{\|w, s\|_{DG}}.
\]

Let us make precise the abstract error bound above for a smooth solution \((\mathbf{u}, p) \in H^{s+1}(\Omega)^3 \times H^s(\Omega), s \geq 1\), on isotropically refined meshes with mesh-size \(h\) with possible hanging nodes and for mixed-order elements where \(\ell = k - 1\).

In this case, the residual \(R_h(\mathbf{u}, p)\) can be bounded (see Proposition 8.1 of Ref. 24) by

\[
R_h(\mathbf{u}, p) \leq \sup_{(w, s) \in \mathbf{W}_h} \left| \int_{\mathcal{F}} \left\{ \nu \nabla \mathbf{u} - T(\nu \nabla \mathbf{u}) \right\} : \mathbf{w} \right| ds + \left| \int_{\mathcal{F}} \left\{ p - T(p) \right\} \mathbf{w} \right| ds \right| \|w, s\|_{DG},
\]

where \(T\) and \(T\) are the \(L^2\)-projections onto \(\sum_h\) and \(Q_h\), respectively. The Cauchy-Schwarz inequality and standard \(hp\)-approximation properties then give

\[
R_h(\mathbf{u}, p) \leq C h^{\min\{s,k\}} \left[ \nu^\frac{1}{2} \|\mathbf{u}\|_{s+1} + \nu^{-\frac{1}{2}} \|q\|_s \right].
\]

Furthermore,

\[
\inf_{(v, q) \in \mathbf{W}_h} \|\mathbf{u} - v, p - q\|_{DG} \leq C h^{\min\{s,k\}} \left[ \nu^\frac{1}{2} \|\mathbf{u}\|_{s+1} + \nu^{-\frac{1}{2}} \|q\|_s \right]
\]

and thus

\[
\|\mathbf{u} - \mathbf{u}_h, p - p_h\|_{DG} \leq C h^{\min\{s,k\}} \left[ \nu^\frac{1}{2} \|\mathbf{u}\|_{s+1} + \nu^{-\frac{1}{2}} \|q\|_s \right].
\]  

(6.9)

This estimate is optimal in the mesh-size \(h\) and suboptimal in \(k\) by one power of \(k\) in the velocity and by a power \(k^{3/2}\) in the pressure, respectively.

Similarly to Sect. 8 of Ref. 24, we obtain a slightly better result on conforming meshes, that is,

\[
\|\mathbf{u} - \mathbf{u}_h, p - p_h\|_{DG} \leq C h^{\min\{s,k\}} \left[ \nu^\frac{1}{2} \|\mathbf{u}\|_{s+1} + \nu^{-\frac{1}{2}} \|q\|_s \right].
\]  

(6.10)

We point out that the a-priori error bounds (6.9) and (6.10) hold verbatim for equal-order elements.
Remark 6.3 We note that the dependence on the polynomial degree $k$ in (6.9) and (6.10) is better than in the $hp$-estimates of Ref. 24 for mixed-order $Q_k - Q_{k-1}$ elements without pressure stabilization, by half an order of $k$ in the velocity and a full order of $k$ in the pressure, respectively.

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