EXPONENTIAL CONVERGENCE FOR hp-VERSION AND SPECTRAL FINITE ELEMENT METHODS FOR ELLIPTIC PROBLEMS IN POLYHEDRA

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Received (Day Month Year)
Revised (Day Month Year)
Accepted (Day Month Year)
Communicated by (xxxxxxxxxx)

We establish exponential convergence of conforming hp-version and spectral finite element methods for second-order, elliptic boundary-value problems with constant coefficients and homogeneous Dirichlet boundary conditions in bounded, axiparallel polyhedra. The source terms are assumed to be piecewise analytic. The conforming hp-approximations are based on e-geometric meshes of mapped, possibly anisotropic hexahedra and on the uniform and isotropic polynomial degree $p \geq 1$. The principal new results are the construction of conforming, patchwise hp-interpolation operators in edge, corner and corner-edge patches which are the three basic building blocks of geometric meshes. In particular, we prove, for each patch type, exponential convergence rates for the $H^1$-norm of the corresponding hp-version (quasi)interpolation errors for functions which belong to a suitable, countably normed space on the patches. The present work extends recent hp-version discontinuous Galerkin approaches to conforming Galerkin finite element methods.

Keywords: hp-FEM, spectral FEM, second-order elliptic problems in polyhedra, exponential convergence

AMS Subject Classification: 65N30
1. Introduction

The \emph{hp}-version of the finite element method (FEM) is a realization of so-called variable-degree, variable knot spline approximations in the context of Galerkin approximations for elliptic partial differential equations; cp. Refs. 6, 13 and the references therein. While in Refs. 6, 13 exponential convergence rates in $L^\infty$-norm were proved for particular singular functions in one space dimension, in Ref. 9 exponential convergence of \emph{hp}-FEM in $H^1$-norm was shown for a model second-order, elliptic boundary-value problem, again with a model singular solution in one space dimension. Subsequently, these concepts were substantially generalized to \emph{hp}-version FEMs for second-order, elliptic boundary-value problems in two space dimensions: on the one hand, rather than for particular singular solutions, in Ref. 10 exponential convergence of \emph{hp}-FEM on geometrically refined triangulations was now proved for solutions belonging to \textit{countably normed, weighted Sobolev spaces}. On the other hand, in Ref. 1 an \textit{elliptic regularity shift theorem} was shown in these spaces.

In recent years, the corresponding elliptic regularity shifts in countably normed, weighted Sobolev spaces in polyhedral domains have been established in Refs. 11, 12, 4, at least for certain classes of second-order, elliptic boundary-value problems in three dimensions. Based on these analytic regularity results, in Ref. 15, exponential convergence in broken $H^1$-norms has been proved for an \emph{hp}-version discontinuous Galerkin (DG) method for second-order, elliptic problems in axiparallel polyhedra. Corresponding exponential convergence results on regular, anisotropic geometric meshes of tetrahedral elements have been announced in Ref. 2.

The purpose of the present paper is to establish exponential convergence of conforming \emph{hp}-version approximations on families of geometrically refined meshes of axiparallel hexahedra, in the setting of Refs. 14, 15. Our proof consists in the construction of $H^1$-conforming piecewise polynomial approximations of \textit{uniform polynomial degree} $p \geq 1$, by modifying our \textit{discontinuous \emph{hp}-version base interpolants} from Ref. 15 with the aid of suitable \textit{polynomial trace lifting operators} in the presence of \textit{irregular} geometric mesh refinements towards corners and edges. This construction is accompanied with the proof that the trace liftings thus constructed do not disrupt the exponential convergence of the base interpolants.

Therefore, the main results of the paper are, for each geometric mesh patch of corner, edge and corner-edge type, the construction and analysis of \emph{hp}-version patch projectors onto spaces of continuous, piecewise polynomials which converge exponentially for solutions belonging to a certain analytic class of functions in each patch type. By Céa’s lemma, this result in conjunction with the analytic regularity in Ref. 4 implies exponential convergence of \emph{hp}-FEM on polyhedra for the model diffusion equation considered in the paper. We emphasize, however, that the \textit{hp-patch approximation results} proved here apply more generally to any solution whose pull-backs into the reference patches belong to one of the weighted analytic classes.

The present \emph{hp}-version consistency error analysis covers, in particular, also spec-
central element methods: there, additional quadrature errors due to elemental underintegration arise, which are not discussed in the present paper. We also mention that conforming \(hp\)-FE spaces have been implemented in Refs. 8, 7.

The outline of this paper is as follows. In Section 2 we present an elliptic model problem in an axiparallel polyhedral domain, and recapitulate the analytic regularity theory of Ref. 4 for solutions with piecewise analytic source terms. Section 3 addresses the construction of the \(hp\)-version subspaces, and it presents and discusses our main result on exponential convergence of conforming \(hp\)-FEM. Section 4 reviews the construction and exponential error bounds of the base \(hp\)-version projector from Ref. 15 which are obtained by tensorization of univariate \(hp\)-projectors. In Section 5, we construct and analyze polynomial trace liftings which preserve exponential convergence estimates for all types of irregular interfaces.

The notation employed throughout this paper is consistent with that in Refs. 14, 15. In particular, we shall frequently use the notations \("\lesssim\"\) or \("\approx\"\) to mean an inequality or an equivalence containing generic positive multiplicative constants which are independent of the local mesh size, the polynomial degree \(p\), the regularity parameters, and the geometric refinement level \(\ell\), but which may depend on the geometric refinement ratio \(\sigma\).

2. Elliptic Model Problem and Analytic Regularity

In this section, we introduce an elliptic model problem in an axiparallel polyhedron and specify the (analytic) regularity of its solutions in terms of countably normed weighted Sobolev spaces. We follow Ref. 4, based on the notations already introduced in Refs. 14, 15.

2.1. Model problem

Let \(\Omega \subset \mathbb{R}^3\) be an open, bounded and axiparallel polyhedron with Lipschitz boundary \(\Gamma = \partial \Omega\) that consists of a finite union of plane faces. We consider the elliptic boundary-value problem

\[
-\nabla \cdot (A \nabla u) = f \quad \text{in } \Omega,
\]

\[
\gamma_0(u) = 0 \quad \text{on } \Gamma,
\]

where \(\gamma_0\) denotes the trace operator on \(\Gamma\). We assume that the diffusion tensor \(A\) is constant and symmetric positive definite. With the standard Sobolev space \(H^1_0(\Omega) = \{v \in H^1(\Omega) : \gamma_0(v) = 0\}\), the variational form of problem (2.1)–(2.2) is to find \(u \in H^1_0(\Omega)\) such that

\[
a(u, v) := \int_\Omega A \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in H^1_0(\Omega).
\]

Under the above assumptions, for every right hand side \(f\) in \(H^{-1}(\Omega)\), the dual space of \(H^1_0(\Omega)\), problem (2.3) admits a unique weak solution \(u \in H^1_0(\Omega)\).
2.2. Subdomains and weights

We denote by $C$ the set of corners $c$, and by $E$ the set of open edges $e$ of $\Omega$. The singular set of $\Omega$ is given by

$$S := \left( \bigcup_{c \in C} c \right) \cup \left( \bigcup_{e \in E} e \right) \subset \Gamma.$$  \hfill (2.4)

For $c \in C$, $e \in E$, and $x \in \Omega$, we define the following distance functions:

$$r_c(x) := \text{dist}(x, c), \quad r_e(x) := \text{dist}(x, e), \quad \rho_{ce}(x) := r_e(x)/r_c(x).$$  \hfill (2.5)

As in Ref. 14, the vertices of $\Omega$ are assumed to be separated. For each corner $c \in C$, we denote by $E_c := \{ e \in E : c \cap \overline{e} \neq \emptyset \}$ the set of all edges which meet at $c$. Similarly, for $e \in E$, the set of corners of $e$ is $C_e := \{ c \in C : c \cap \overline{e} \neq \emptyset \}$. Then, for $\varepsilon > 0$, $c \in C$, $e \in E$ respectively, we define the neighborhoods

$$\omega_c := \{ x \in \Omega : r_c(x) < \varepsilon \land \rho_{ce}(x) > \varepsilon \quad \forall e \in E_c \},$$

$$\omega_e := \{ x \in \Omega : r_e(x) < \varepsilon \land r_c(x) > \varepsilon \quad \forall c \in C_e \},$$

$$\omega_{ce} := \{ x \in \Omega : r_e(x) < \varepsilon \land \rho_{ce}(x) < \varepsilon \}.$$  \hfill (2.6)

By choosing $\varepsilon > 0$ sufficiently small, the domain $\Omega$ can be partitioned into four disjoint subdomains, $\Omega = \Omega_C \cup \Omega_E \cup \Omega_{CE} \cup \Omega_0$, referred to as corner, edge, corner-edge and interior neighborhoods of $\Omega$, respectively, where

$$\Omega_C = \bigcup_{c \in C} \omega_c, \quad \Omega_E = \bigcup_{e \in E} \omega_e, \quad \Omega_{CE} = \bigcup_{c \in C} \bigcup_{e \in E} \omega_{ce}, \quad \Omega_0 := \Omega \setminus \Omega_C \cup \Omega_E \cup \Omega_{CE}.$$  \hfill (2.7)

2.3. Weighted Sobolev spaces

To each $c \in C$ and $e \in E$, we associate a corner and an edge exponent $\beta_c, \beta_e \in \mathbb{R}$, respectively, and introduce the vector $\beta := \{ \beta_c \}_{c \in C} \cup \{ \beta_e \}_{e \in E} \in \mathbb{R}^{C+E}$. Inequalities of the form $\beta < 1$ and expressions like $\beta \pm s$ are to be understood componentwise.

For $e \in E$ or $e \in E_c$ with $c \in C$, we choose local coordinate systems in $\omega_e$ and $\omega_{ce}$ such that the edge $e$ corresponds to the direction $(0, 0, 1)$. We indicate quantities perpendicular to $e$ by $(\cdot)^\perp$, and quantities parallel to $e$ by $(\cdot)^\parallel$. In particular, if $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ is a multi-index of order $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, then we use the notation $\alpha = (\alpha^\perp, \alpha^\parallel)$ with $\alpha^\perp = (\alpha_1, \alpha_2)$ and $\alpha^\parallel = \alpha_3$. We further write the partial derivative operator $D^\alpha$ as $D^\alpha = D^\alpha_{\perp} D^\alpha_{\parallel}$, where $D^\alpha_{\perp}$ and $D^\alpha_{\parallel}$ signify the derivative operators in perpendicular and parallel directions. For $k \in \mathbb{N}_0$, we then introduce the weighted semi-norm

$$\|u\|^2_{H_k^\beta(\Omega)} := \sum_{|\alpha| = k} \left\{ \|D^\alpha u\|^2_{L^2(\Omega_0)} + \sum_{e \in E} \|r_e^{\beta_e + |\alpha^\perp|} D^\alpha u\|^2_{L^2(\omega_e)} \right\} + \sum_{e \in E} \|r_e^{\beta_e + |\alpha^\perp|} D^\alpha u\|^2_{L^2(\omega_e)}.$$  \hfill (2.8)
The norm $\| \cdot \|_{M^m_\beta(\Omega)}$ is defined by $\|u\|^2_{M^m_\beta(\Omega)} = \sum_{k=0}^{m} |u|_{k,M^m_\beta(\Omega)}^2$, and the weighted Sobolev space $M^m_\beta(\Omega)$ is obtained as the closure of $C^\infty_0(\Omega)$ with respect to the norm $\| \cdot \|_{M^m_\beta(\Omega)}$. For an open, non-empty subset $D \subseteq \Omega$, we denote by $\| \cdot \|_{M^m_\beta(D)}$ and $\| \cdot \|_{M^m_\beta(D)}$ the above semi-norm and norm, respectively, with all domains of integration replaced by their intersections with $D \subseteq \Omega$.

2.4. Analytic regularity of variational solutions

We adopt the following classes of analytic functions from Ref. 4 (Section 6.2).

**Definition 2.1.** For subdomains $D \subseteq \Omega$ the space $A_\beta(D)$ consists of all functions $u$ such that $u \in M^m_\beta(D)$ for all $k \geq 0$, and such that there exists a constant $C_u > 0$ (independent of $D$) with the property that

$$|u|_{k,M^m_\beta(D)} \leq C_u^{k+1} \Gamma(k+1), \quad \forall k \in \mathbb{N}_0,$$

where $\Gamma$ is the standard Gamma function satisfying $\Gamma(k+1) = k!$ for $k \in \mathbb{N}_0$.

From Ref. 4 (Corollary 7.1), the following analytic regularity result holds.

**Proposition 2.1.** There are bounds $b_c, b_e > 0$ (depending on $\Omega$ and the coefficient matrix $A$) such that, for weight exponent vectors $b$ satisfying

$$0 \leq b_c < b_e, \quad 0 \leq b_c < b_e, \quad c \in \mathcal{C}, \quad e \in \mathcal{E}, \quad (2.10)$$

the weak solution $u \in H^1_0(\Omega)$ in (2.3) of the Dirichlet problem (2.1)-(2.2) satisfies:

$$f \in A_{1-b_c}(\Omega) \implies u \in A_{-1-b_e}(\Omega). \quad (2.11)$$

**Remark 2.1.** As in Ref. 15 (Remark 2.2), we exclude the limit cases $b_c = b_e = 0$, and assume without loss of generality that in (2.10) there holds

$$0 < b_c < 1, \quad 0 < b_e < 1, \quad c \in \mathcal{C}, \quad e \in \mathcal{E}. \quad (2.12)$$

3. $hp$-Version Discretization and Exponential Convergence

In this section, we adjust the construction of $hp$-version finite element spaces on geometric mesh families as presented in Refs. 14, 15 from the discontinuous to the continuous Galerkin framework. Then, we introduce conforming $hp$-version finite element approximations, state our main exponential convergence result (Theorem 3.3), and outline the structure of its proof.

3.1. Geometric meshes

We start the construction of geometric meshes from a coarse regular and quasiuniform patch mesh $\mathcal{M}^0 = \{ Q_p \}_{p=1}^P$, which partitions $\Omega$ into $\mathcal{P}$ convex and axiparallel hexahedra also referred to as patches. Throughout, we shall assume that the initial mesh $\mathcal{M}^0$ is sufficiently fine so that an element $K \in \mathcal{M}^0$ has non-trivial intersection
with at most one corner \( c \in \mathcal{C} \), and either none, one or several edges \( e \in \mathcal{E} \) meeting in \( c \). Each of these patch subdomains \( Q_p \in \mathcal{M}^0 \) is the image under an affine mapping \( G_p \) of the reference patch domain \( \tilde{Q} = (-1,1)^3 \), i.e., \( Q_p = G_p(\tilde{Q}) \).

![Fig. 1. Three geometric reference mesh patches on \( \tilde{Q} \) with subdivision ratio \( \sigma = 0.5 \): corner patch \( \mathcal{M}^\ell_e \) with isotropic geometric refinement towards the corner \( c \) (left), edge patch \( \mathcal{M}^\ell_e \) with anisotropic geometric refinement towards the edge \( e \) (center), and corner-edge patch \( \mathcal{M}^\ell_{ce} \) with geometric refinement towards the corner-edge pair \( c e \) (right). The singular supports \( c, e, ce \) are shown in bold face.](image)

With each patch \( Q_p \in \mathcal{M}^0 \), we associate one of four types of geometric reference mesh patches \( \mathcal{M}_p \) on \( \tilde{Q} \), as constructed in Ref. 14 (Section 3.3) in terms of four \( hp \)-extensions (Ex1)--(Ex4). That is,

\[
\mathcal{M}_p \in \mathcal{RP} := \{ \mathcal{M}^\ell_e, \mathcal{M}^\ell_e, \mathcal{M}^\ell_{ce}, \mathcal{M}^\ell_{int} \} = \{ \mathcal{M}^\ell_{\sigma,t} \}_{t \in \{ e, ce, int \}}.
\]

More specifically, whenever \( Q_p \) abuts at the singular set \( S \), we take \( \mathcal{M}_p \) as a suitably rotated and oriented version of the geometrically refined reference mesh patches shown in Figure 1 and denoted by \( \mathcal{M}^\ell_e \) (corner patch), \( \mathcal{M}^\ell_e \) (edge patch), and \( \mathcal{M}^\ell_{ce} \) (corner-edge patch), respectively. We implicitly allow for simultaneous geometric refinements towards several edges in the corner-edge patch \( \mathcal{M}^\ell_{ce} \), which corresponds to an overlap of at most three rotated versions of the basic corner-edge patch; see also Figures 10 and 11 ahead. The geometric refinements in these reference patches are characterized by (i) a fixed parameter \( \sigma \in (0,1) \) defining the subdivision ratio of the geometric refinements and (ii) the index \( \ell \) defining the number of refinements. For interior patches \( Q_p \in \mathcal{M}^0 \), which have empty intersection with \( S \), we assign to \( \mathcal{M}_p \) a geometric reference mesh patch \( \mathcal{M}^\ell_{int} \) on \( \tilde{Q} \), which comprises only finitely many regular refinements and does not introduce irregular faces within \( \tilde{Q} \). In the refinement process (i.e., as \( \ell \to \infty \)), the reference mesh \( \mathcal{M}^\ell_{int} \) is left unchanged and is in fact independent of \( \ell \).

The geometric reference mesh patch \( \mathcal{M}_p \in \mathcal{RP} \) introduces the corresponding patch partition \( \mathcal{M}_p = G_p(\mathcal{M}_p) := \{ K : K = G_p(\tilde{K}) , \tilde{K} \in \mathcal{M}_p \} \) on \( Q_p \), where \( G_p \) is the affine patch map. To ensure continuity across mesh patches \( \mathcal{M}_p \), we shall always work under the following inter-patch compatibility hypothesis.

**Assumption 3.1.** For \( p \neq p' \), let \( Q_p, Q_{p'} \in \mathcal{M}^0 \) be two distinct patches with
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non-empty intersection $\Gamma_{pp'}, := \overline{Q_p} \cap \overline{Q_{p'}} \neq \emptyset$. Then the parametrizations induced by patch maps on patch interfaces $\Gamma_{pp'}$ are assumed to coincide “from either side”:

$G_p \circ (G^{-1}_p |_{\Gamma_{pp'}}) = G_{p'} \circ (G^{-1}_{p'} |_{\Gamma_{pp'}})$. In addition, the mesh patches $\mathcal{M}_p, \mathcal{M}_{p'}$ are assumed to coincide on $\Gamma_{pp'}$.

Hence, for fixed parameters $\sigma \in (0,1)$ and $\ell \in \mathbb{N}$, a $\sigma$-geometric mesh on $\Omega$ is now given by the disjoint union

$$\mathcal{M} = \mathcal{M}_{\sigma}^{(\ell)} := \bigcup_{p=1}^{\Psi} \mathcal{M}_p. \quad (3.2)$$

By construction, each axiparallel element $K \in \mathcal{M}$ is the image of the reference cube $\tilde{K}$ under an element mapping $K = \Phi_K(\tilde{K})$, where $\Phi_K$ is the composition of the corresponding patch map $G_p$ with an anisotropic dilation-translation from $\tilde{K}$ to $K$. To achieve a proper geometric refinement towards corners and edges of $\Omega$ without violating Assumption 3.1, the geometric refinements $\mathcal{M}_p$ in the patches $Q_p$ have to be suitably selected and oriented. For a fixed subdivision ratio $\sigma \in (0,1)$, we call the sequence $\mathcal{M}_p = \{\mathcal{M}_p^{(\ell)}\}_{\ell \geq 1}$ of geometric meshes a $\sigma$-geometric mesh family; see Ref. 14 (Definition 3.4). As before, we shall refer to the index $\ell$ as refinement level.

We denote the sets of all interior and boundary edges $e$ of a geometric mesh $\mathcal{M}$ by $\mathcal{E}_i(\mathcal{M})$ and $\mathcal{E}_b(\mathcal{M})$, respectively, and set $\mathcal{E}(\mathcal{M}) := \mathcal{E}_i(\mathcal{M}) \cup \mathcal{E}_b(\mathcal{M})$. We denote by $e_{K,K'} := \interior(\partial K \cap \partial K')$ an edge shared by $K$ and $K'$. Similarly, we write $\mathcal{F}_i(\mathcal{M})$ and $\mathcal{F}_b(\mathcal{M})$ for the sets of interior and boundary faces $f$ of $\mathcal{M}$, respectively, and define $\mathcal{F}(\mathcal{M}) := \mathcal{F}_i(\mathcal{M}) \cup \mathcal{F}_b(\mathcal{M})$. Again, we write $f_{K,K'} := \interior(\partial K \cap \partial K')$ for the interior face shared by $K$ and $K'$. For a piecewise smooth function $v$, we denote the jump of $v$ over the face $f_{K,K'}$ by

$$[v]_{f_{K,K'}} := v|_K - v|_{K'}. \quad (3.3)$$

Moreover, for an element $K$, we denote by $\mathcal{E}_K$ the set of its elemental edges, and by $\mathcal{F}_K$ the set of its elemental faces. We call an edge $e$ \textit{regular} if $e$ is an entire elemental face of all elements $K$ sharing it (i.e., if $\Sigma \cap \tilde{K} = \emptyset$, then $e \in \mathcal{E}_K$). Otherwise $e$ is called \textit{irregular}. Analogously, a face $f = f_{K,K'}$ is called regular if $f$ is an elemental face of both $K$ and $K'$ (i.e., $f \in \mathcal{F}_K$ and $f \in \mathcal{F}_{K'}$); otherwise it is irregular. We shall always assume that boundary edges or faces belong to exactly one boundary plane of $\Gamma$. For the continuity of finite element functions across elements, we impose the following assumption.

\textbf{Assumption 3.2.} Under Assumption 3.1, for any two, distinct elements $K, K' \in \mathcal{M}$ which share either a common edge $e_{K,K'}$ or an interior face $f_{K,K'}$, the traces of the elemental polynomial spaces on $e_{K,K'}$ or $f_{K,K'}$ in local coordinates (induced by the corresponding patch maps) coincide.

Following Ref. 14, we may partition a geometric mesh $\mathcal{M}_{\sigma}^{(\ell)}$ into interior ele-
ments $\mathcal{D}_e^\ell$ away from $S$ and into the terminal layer elements $\mathcal{T}_e^\ell$ at $S$:

$$\mathcal{M}_e^{(\ell)} := \mathcal{D}_e^\ell \cup \mathcal{T}_e^\ell,$$

(3.4)

with $\mathcal{D}_e^\ell := \{K \in \mathcal{M}_e^{(\ell)} : \overline{K} \cap S = \emptyset\}$ and $\mathcal{T}_e^\ell := \{K \in \mathcal{M}_e^{(\ell)} : \overline{K} \cap S \neq \emptyset\}$. The interior mesh $\mathcal{D}_e^\ell$ can be further disjointly partitioned into $\ell$ mesh layers of the form

$$\mathcal{D}_e^\ell \cup \ldots \cup \mathcal{D}_e^{\ell-1},$$

(3.5)

where mesh layer $0 \leq \ell' \leq \ell - 1$ consists of a group $\mathcal{L}_e^{\ell'}$ of elements with identical scaling properties; cp. Ref. 15 (Section 3).

Next, we establish some anisotropic scaling properties. For $K \in \mathcal{M}_e^{(\ell)}$, we set $h_K := \text{diam}(K)$. As in Refs. 14, 15, for possibly anisotropic edge-patch and corner-edge patch elements $K$, we denote by $h_K^\perp$ and $h_K^\parallel$ the elemental diameters of $K$ transversal respectively parallel to the singular edge $e \in \mathcal{E}$ situated nearest to $K$, defined as the corresponding quantities over the axiparallel element $\tilde{K} = G_p^{-1}(K)$. If $K \in \mathcal{D}_e^{\ell'}$, these quantities are related to the relative distances to the sets $\mathcal{C}$ and $\mathcal{E}$; cp. Ref. 15 (Proposition 3.2). Corner-patch and interior-patch elements are isotropic, with $h_K^\perp \approx h_K^\parallel \approx h_K$. We further denote by $h_K^{\perp,f}$ the height of $K \in \mathcal{M}_e^{(\ell)}$ in direction perpendicular to $F \in \mathcal{F}_K$, also defined as the corresponding height in the axiparallel element $\tilde{K} = G_p^{-1}(K)$. As in Ref. 15 (Section 5.1.4), we may assume without loss of generality that $K \in \mathcal{M}_e^{(\ell)}$ can be written in the form

$$K = K^\perp \times K^\parallel = (0, h_K^\parallel)^2 \times (0, h_K^\parallel).$$

(3.6)

**Lemma 3.1.** Let $K$ be an axiparallel element of the form (3.6) and $\Phi_K : \tilde{K} \to K$ the element transformation. For $v : K \to \mathbb{R}$ and $\tilde{v} = v \circ \Phi_K$, we have the scalings:

\begin{itemize}
  \item [(i)] \(\|v\|_{L^2(K)} \approx (h_K^\parallel)^2 h_K^\parallel \|	ilde{v}\|_{L^2(\tilde{K})}\).
  \item [(ii)] If $h_K^\parallel \leq h_K$, then $\|
abla v\|^2_{L^2(K)} \lesssim h_K^\parallel 2\|
abla \tilde{v}\|^2_{L^2(\tilde{K})}$.
  \item [(iii)] If $f \in \mathcal{F}_K$ is an elemental face of $K$ with $h_{K,f}^\parallel \approx h_K^\parallel$ and $\tilde{f}$ the corresponding reference face of $\tilde{K}$, then $\|\tilde{v}\|^2_{L^2(\tilde{f})} \approx \left(\frac{h_{K,f}^\parallel}{h_K^\parallel}\right)^{-1} \|v\|^2_{L^2(f)}$.
\end{itemize}

**Proof.** The following more general scaling property was established in Ref. 15 (Equation (5.11)):

$$\|D^\alpha \tilde{D}^\alpha v\|^2_{L^2(\tilde{K})} = \left(\frac{h_{K}^\parallel}{2}\right)^{2\alpha} \|D^\alpha v\|^2_{L^2(K)}.$$

(3.7)

The $L^2$-norm scaling in item (i) is an immediate consequence of (3.7). Similarly, we see that

$$\|D_{\perp} v\|^2_{L^2(K)} \approx h_K^\parallel \|D_{\perp} \tilde{v}\|^2_{L^2(\tilde{K})}, \quad \|D_{\parallel} v\|^2_{L^2(K)} \approx (h_K^\parallel)^2 (h_K^\parallel)^{-1} \|D_{\parallel} \tilde{v}\|^2_{L^2(\tilde{K})}.$$ 

Hence, if $h_K^\parallel \leq h_K$, item (ii) follows. To show item (iii), we note that $h_{K,f}^\parallel \approx h_K^\parallel$ implies that $f$ can be written in the form $f = (0, h_K^\parallel) \times (0, h_K^\parallel)$. Hence, item (iii) follows from a similar $L^2$-norm scaling argument. \(\square\)
Finally, we establish an anisotropic jump estimate. To state it, we define the weighted $H^1(K)$-norm:

$$N_K[v]^2 := (h_{\text{min},K}^{-1})^2 \|v\|_{L^2(K)}^2 + \|\nabla v\|_{L^2(K)}^2, \quad K \in \mathcal{M}_\sigma^{(t)}.$$  \hfill (3.8)

with $h_{\text{min},K} := \min_{F \in \mathcal{F}_K} \{h_F^{-1}\}$. We then consider an interior face $f_{K,K'}$ parallel to a singular edge $e$ and shared by two axiparallel and possibly non-matching hexahedra $K, K'$ of the form $K = K^- \times (0, i h)$ and $K' = (K')^- \times (0, i h)$, cp. (3.6). The elements $K^-, (K')^-$ are shape-regular rectangles with $\text{diam}(K^-) \simeq \text{diam}((K')^-) \simeq h^-$, for $h^- \lesssim h$.

**Lemma 3.2.** In the setting above and for a piecewise smooth function $v$, we have

$$(h^-)^{-1} \|[v]_{f_{K,K'}}\|_{L^2(f_{K,K'})}^2 \lesssim N_K[v]^2 + N_{K'}[v]^2. \quad \hfill (3.9)$$

**Proof.** We have $h_{K,K'}^{-1} \simeq h_{\text{min},K}^{-1} h_{\text{min},K'}^{-1} \simeq h_{\text{min},K}^{-1} h_{\text{min},K'}^{-1} \simeq h^-$, Hence, applying the anisotropic trace inequality from Ref. 14 (Lemma 4.2 with $t = 2$) yields

$$\|[v]_{f_{K,K'}}\|_{L^2(f_{K,K'})}^2 \lesssim \|v|_{K}^2\|_{L^2(f_{K,K'})}^2 + \|v|_{K'}^2\|_{L^2(f_{K,K'})}^2$$

$$\lesssim (h^-)^{-1} \left(\|v\|_{L^2(K)}^2 + \|v\|_{L^2(K')}^2\right) + h^- \left(\|\nabla v\|_{L^2(K)}^2 + \|\nabla v\|_{L^2(K')}^2\right).$$

The bound (3.9) follows. \hfill $\square$

### 3.2. $hp$-Version discretizations

The conforming finite element spaces to be considered here are based on the uniform and isotropic polynomial degree $p \geq 1$ (throughout $K \in \mathcal{M}$). For a geometric mesh $\mathcal{M}$ satisfying Assumptions 3.1, 3.2, we introduce two $hp$-version finite element spaces

$$V_p(\mathcal{M}) := \left\{ v \in H^1(\Omega) : v|_K \in Q_p(K), \ K \in \mathcal{M} \right\},$$

$$V^0_p(\mathcal{M}) := V_p(\mathcal{M}) \cap H^1_0(\Omega). \hfill (3.10)$$

Here, the local polynomial approximation space $Q_p(K)$ is defined as follows. First, on the reference element $\tilde{K}$, we introduce the tensor-product polynomial space $Q_{\tilde{p}}(\tilde{K}) := \text{span} \{ \tilde{x}^\alpha : \alpha_i \leq p, 1 \leq i \leq 3 \}$. Then, for an hexahedral element $K \in \mathcal{M}$ with elemental mapping $\Phi_K : \tilde{K} \rightarrow K$, we set $Q_p(K) := \left\{ v \in L^2(K) : v|_K \circ \Phi_K \in Q_{\tilde{p}}(\tilde{K}) \right\}$. As compared to discontinuous spaces considered Refs. 14, 15, the spaces in (3.10) now feature interelement continuity and essential boundary conditions in the presence of geometric mesh refinements.

Under Assumptions 3.1, 3.2, let $\mathfrak{M}_\sigma = \{ \mathcal{M}_\sigma^{(t)} \}_{t \geq 1}$ be a $\sigma$-geometric mesh family and $\mu > 0$ a proportionality parameter. Then we consider the sequence $\{V^\ell_p(\mathcal{M}_\sigma^{(t)})\}_{t \geq 1}$ of $hp$-version spaces of uniform polynomial degree $p_\ell$ given by

$$V^\ell_p := V^0_p(\mathcal{M}_\sigma^{(t)}), \quad \text{with} \ p_\ell := \max\{3, [\mu \ell]\}, \quad \ell \geq 1. \hfill (3.11)$$

Here we note that, as in Ref. 15, we shall always work under the (purely technical) assumption that $p_\ell \geq 3$; cp. Lemma 4.1 below.
Remark 3.1. Due to the occurrence of irregular faces and edges in the geometric refinements, it is a-priori not clear that the spaces (3.10), (3.11) are well-defined. That these definitions, indeed, define proper linear subspaces will follow from our construction of polynomial trace liftings in Section 5 ahead.

With the definition (3.11) of the \( h p \)-FE spaces in place, the \( h p \)-version Galerkin discretization of the variational formulation (2.3) reads as usual:

\[
\eta^\ell \in V^\ell_\sigma: \ F(u^\ell_\sigma, v) = \int_\Omega f v dx \quad \forall v \in V^\ell_\sigma. \tag{3.12}
\]

For \( \ell \geq 1 \), the discrete problem (3.12) has a unique solution \( u^\ell_\sigma \) which is quasi-optimal: there exists a constant \( C > 0 \) (depending only on the domain \( \Omega \) and the coefficient matrix \( A \)) such that for all parameters \( \sigma, \ell \) there holds

\[
\|u - u^\ell_\sigma\|_{H^1(\Omega)} \leq C \inf_{v \in V^\ell_\sigma} \|u - v\|_{H^1(\Omega)}. \tag{3.13}
\]

3.3. Exponential convergence

Our main convergence result is as follows.

Theorem 3.3. Let \( \mathcal{M}_\sigma = \{\mathcal{M}^{(\ell)}_\sigma\}_{\ell \geq 1} \) be a family of \( \sigma \)-geometric meshes on \( \Omega \), and consider the \( h p \)-version discretizations based on the sequence \( V^\ell_\sigma \) of subspaces defined in (3.11). Then, for weight exponents \( b \) as in (2.12) and \( \ell \geq 1 \), there exist projectors \( \Pi^\ell : V^{6-\beta}_M(\Omega) \to V^\ell_\sigma \) such that for \( u \in A^{1-b}(\Omega) \subset M^{6-\beta}_M(\Omega) \) we have the bound

\[
\|u - \Pi^\ell u\|_{H^1(\Omega)} \leq C \exp (-b\ell). \tag{3.14}
\]

The constants \( b, C > 0 \) are independent of \( \ell \), but depend on the subdivision ratio \( \sigma \), the patch mesh \( \mathcal{M}^0 \) with its associated patch maps, the weight exponents \( b > 0 \) (see Remark 2.1), the proportionality constant \( \mu > 0 \) in (3.11), and on the constant \( C_u \) in the analytic regularity estimate (2.9).

In particular, if the source term \( f \) in the boundary-value problem (2.1)–(2.2) belongs to \( A^{1-b}(\Omega) \) with weights as in (2.12) (and hence the solution \( u \) is in \( A^{1-b}(\Omega) \) by Proposition 2.1), then, as \( \ell \to \infty \), the \( h p \)-version approximations \( u^\ell_\sigma \) in (3.12) converge exponentially

\[
\|u - u^\ell_\sigma\|_{H^1(\Omega)} \leq C \exp \left(-b\sqrt{\ell^p}\right), \tag{3.15}
\]

where the constants \( b, C > 0 \) are independent of \( N = \dim(V^\ell_\sigma) \), the number of degrees of freedom of the \( h p \)-FE discretization.

3.4. Outline of proof

We first note that the error bound (3.15) is an immediate consequence of the quasi-optimality property (3.13) and the exponential consistency bound (3.14) (noting that \( N \approx \ell^p \)). Therefore, the proof of Theorem 3.3 will follow from the construction
and exponential convergence estimates for the \(hp\)-version (quasi)interpolants \(\Pi^t\). These estimates are of independent interest, and the rest of the paper is devoted to their proof, which is structured as follows.

The \(hp\)-version projector \(\Pi^t\) in (3.14) will be assembled from corresponding patch projectors \(\Pi^t_p\) for \(1 \leq p \leq \mathfrak{P}\). To do so, consider the mesh patch \(\mathcal{M}_p = G_p(\mathcal{M}_p)\) on \(Q_p\). Then, with the geometric reference mesh patch \(\mathcal{M}_p\), we associate a reference patch projector \(\Pi^t_p\) on \(\mathcal{M}_p\), which, in accordance with the four types of geometric refinements chosen for \(\mathcal{M}_p\) in (3.1), is taken as one of four types of reference patch projectors

\[
\Pi^t \text{ on } \mathcal{M}^t_{\sigma}, \quad \sigma \in \{c, e, ce, \text{int}\}. \tag{3.16}
\]

On the physical patch \(\mathcal{M}_p\), the patch projector \(\Pi^t_p\) is then defined via

\[
(\Pi^t_p u)|_{Q_p} \circ G_p = \Pi^t_p (u|_{Q_p} \circ G_p). \tag{3.17}
\]

The inter-patch continuity of the projector \(\Pi^t\) defined patchwise as \(\Pi^t|_{Q_p} = \Pi^t_p\) will follow from Assumptions 3.1, 3.2.

Then the proof of (3.14) proceeds by bounding the \(H^1\)-norms of

\[
\eta_p := u|_{Q_p} - \Pi^t_p u|_{Q_p}, \quad p = 1, \ldots, \mathfrak{P}, \tag{3.18}
\]

respectively the pull-backs \(\tilde{\eta}_p\) to the reference patch \(\tilde{Q}\) given by

\[
\tilde{\eta}_p := \tilde{u}_p - \tilde{\Pi}^t_p \tilde{u}_p, \quad p = 1, \ldots, \mathfrak{P}, \tag{3.19}
\]

where \(\tilde{u}_p = u|_{Q_p} \circ G_p\) is the pull-back of \(u|_{Q_p}\) to the reference patch \(\tilde{Q}\). For a finite set \(D\) of axiparallel elements \(K\), we introduce the quantity

\[
\Upsilon_D[v] := \sum_{K \in D} N_K[v]^2, \tag{3.20}
\]

with \(N_K[v]\) defined in (3.8).

**Lemma 3.3.** For \(1 \leq p \leq \mathfrak{P}\), there holds \(\Upsilon_{\mathcal{M}_p}[\eta_p] \simeq \Upsilon_{\mathcal{M}_p}[^t\eta_p]\).

**Proof.** This follows from the construction of the patch mesh \(\mathcal{M}^0\); cp. also the boundedness properties of the patch maps in Ref. 14 (Section 3.1).

By employing (3.18), (3.19) and Lemma 3.3, the approximation error in (3.14) can be bounded by

\[
\|u - \Pi^t u\|_{H^1(\Omega)}^2 \leq \sum_{p=1}^{\mathfrak{P}} \Upsilon_{\mathcal{M}_p}[\eta_p] \lesssim \sum_{p=1}^{\mathfrak{P}} \Upsilon_{\mathcal{M}_p}[\tilde{\eta}_p]. \tag{3.21}
\]

Then, we notice that, up to rotation and orientation, there are four types of reference mesh patches in \(\mathcal{R}\mathcal{P}\) in (3.1). Hence, to bound the right-hand side of (3.21), it is enough to provide error estimates for the four reference cases and we have

\[
\|u - \Pi^t u\|_{H^1(\Omega)}^2 \leq \sum_{t \in \{c, e, ce, \text{int}\}} \Upsilon_{\mathcal{M}^t_{\sigma}}[\tilde{\eta}], \tag{3.22}
\]
with \( \tilde{\eta} := \tilde{u} - \tilde{\Pi}^{t,t}_{b} \tilde{u} \) for the pull-back \( \tilde{u} \) of \( u \) to \( \tilde{Q} \). We observe that, due to the patch maps being affine with (up to rotations and reflections) diagonal Jacobian, the analytic regularity (2.11) of solutions \( u \in H^{1}_{0}(\Omega) \) to (2.1)–(2.2) is preserved under pull-back into the reference patch coordinates on \( \tilde{Q} \). In particular, the pull-back \( \tilde{u} \) of \( u \) to \( \tilde{Q} \) belongs to an analytic regularity reference class \( A_{t}(\tilde{Q}) \) with weighting depending on the type \( t \in \{c, e, ce, int\} \). As in Section 2.2, these reference classes are defined in terms of correspondingly weighted reference semi-norms \( | \cdot |_{M_{t}^{\infty}(\tilde{Q})} \) and spaces \( M_{t}^{\infty}(\tilde{Q}) \), as will be detailed in (4.11), (4.12), (4.13) and (4.14) for any type \( t \in \{c, e, ce, int\} \).

To define the reference patch projectors with exponential error bounds (3.14) on the geometric reference mesh patches \( \tilde{M}_{b}^{t,t} \), we proceed in two steps: first, we introduce base \( hp \)-projectors

\[
\tilde{\pi}^{t,t}_{b} \text{ on } \tilde{M}_{b}^{t,t}, \quad t \in \{c, e, ce, int\},
\]

with exponential approximation bounds under the analytic regularity property in (2.9). As base projectors, we choose the non-conforming and tensorized projectors constructed in Ref. 15 and well-defined for \( \tilde{u} \in M_{b}^{0}(\tilde{Q}) \). The exponential convergence estimates in broken Sobolev norms established in Ref. 15 (Section 5) apply directly on each mesh patch:

\[
t \in \{c, e, ce, int\}, \quad \tilde{u} \in A_{t}(\tilde{Q}) : \quad \Upsilon_{\tilde{M}_{b}^{t,t}}[\tilde{u} - \tilde{\pi}^{t,t}_{b} \tilde{u}] \leq C \exp(-2b\ell),
\]

with constants \( b, C > 0 \) independent of \( \ell \).

The base approximations \( \tilde{\pi}^{t,t}_{b} \) are nodally exact, continuous over matching faces and regular vertices and satisfy the homogeneous essential boundary conditions exactly (on corresponding patch boundary faces). However, the base approximations \( \tilde{\pi}^{t,t}_{b} \) are in general discontinuous across irregular inter-element faces, edges and vertices. To ensure inner-patch continuity (necessary for \( H^{1} \)-conformity) in geometrically refined patches, the next step of our proof therefore consists in constructing jump lifting operators \( \tilde{\mathcal{L}}_{t} \), \( t \in \{c, e, ce\} \), which remove the polynomial jumps in the base \( hp \)-projectors while preserving their exponential convergence estimates and the essential boundary conditions. These polynomial jump-liftings will be introduced and their stability will be analyzed in Section 5. The resulting reference patch \( hp \)-projectors

\[
\tilde{\Pi}^{t,t} := \tilde{\pi}^{t,t}_{b} + \tilde{\mathcal{L}}_{t}, \quad t \in \{c, e, ce\},
\]

then yield continuous, piecewise polynomial approximations, without disrupting the exponential convergence bounds in (3.24). In fact, we establish the following stability estimate, separately for each reference patch.

**Proposition 3.1.** For \( t \in \{c, e, ce, int\} \) and \( \tilde{u} \in M_{b}^{0}(\tilde{Q}) \), we let \( \tilde{\eta}_{t} = \tilde{u} - \tilde{\Pi}^{t,t}_{b} \tilde{u} \) and \( \tilde{\eta}_{b,t} = \tilde{u} - \tilde{\pi}^{t,t}_{b} \tilde{u} \). Then we have

\[
\Upsilon_{\tilde{M}_{b}^{t,t}}[\tilde{\eta}_{t}] \lesssim p^{18} \Upsilon_{\tilde{M}_{b}^{t,t}}[\tilde{\eta}_{b,t}].
\]
for any $t \in \{c, e, ce, \text{int}\}$.

We now conclude the proof of the bound (3.14). The constructions of the reference patch $hp$-projectors lead to a family $\{\Pi^t\}_{t \geq 1}$ of globally conforming, piecewise polynomial and bounded $hp$-version (quasi)interpolants $\Pi^t : M^k_{1-h}(\Omega) \to V^t$. They satisfy the homogeneous essential boundary conditions, and, by (3.22) and Proposition 3.1, converge at the same rates as the base projectors $\pi^t_{h,b}$ on each of the reference mesh patches $M^t_{1-h}$, up to an algebraic loss in the polynomial degree $p$:

$$
\|u - \Pi^t u\|_{H^1(\Omega)}^2 \lesssim p^8 \sum_{t \in \{c, e, ce, \text{int}\}} Y_{\pi^t_{h,b}}[\eta_{b,t}].
$$

As before, for $u \in A_{-1-h}(\Omega)$, the pull-back $\tilde{u}$ of $u|_{Q_p}$ to $\widetilde{Q}$ belongs to one of the analytic reference classes $A_t(\widetilde{Q})$ of type $t \in \{c, e, ce, \text{int}\}$. The proof of (3.14) then follows from (3.27) and the exponential convergence rates of the base interpolants in (3.24). The algebraic loss in $p$ in (3.27) is absorbed by suitably adjusting the constants $b,C$ in the exponential convergence bounds.

**Remark 3.2.** The exponential error bounds for the $hp$-base interpolants in this section are based on the patchwise analytic regularity assumptions in (3.24) of the solution $\tilde{u}$ in local, patch coordinates, which are satisfied in the axi-parallel case considered here. Our construction of exponentially consistent, $H^1$-conforming $hp$-interpolants can be readily extended to curvilinear patches. The exponential convergence bounds in Theorem 3.3 hold, provided that the pull-backs $\tilde{u}$ of the solution $u$ on the patches belongs to one of the analytic regularity reference classes and the patch maps satisfy Assumptions 3.1, 3.2.

**Remark 3.3.** The relatively large, algebraic loss in $p$ in (3.26), (3.27) is an upper bound. It is a consequence of our trace liftings being taken as (bi)linear functions, rather than as polynomials of degree $p$ (which is compatible with the constant polynomial degree $p$). By using polynomial trace-liftings with “minimal energy”, as constructed in Ref. 17 (Lemma 9.1) or Ref. 18, these exponents can be reduced.

It remains to review the definitions of the base projectors in (3.23) and the bounds (3.24) from Ref. 15 (Section 5). This will be done in Section 4. Finally, in Section 5, we present the construction and analysis of the polynomial jump liftings in (3.25) and prove Proposition 3.1.

### 4. Base Projectors and Exponential Convergence

In this section, we specify the non-conforming and tensorized $hp$-version base projectors in (3.23) and review their exponential convergence properties (3.24).

#### 4.1. Tensorization of univariate $hp$-projectors

We begin by introducing the univariate $hp$-approximation operators from Ref. 5. To that end, let $\hat{I} = (-1,1)$ denote the unit interval. For $p \geq 0$, we let $\mathbb{P}_p(\hat{I})$ be
the space of univariate polynomials on \( \tilde{T} \) of degree less or equal than \( p \). We denote by \( \pi_{p,0} : L^2(\tilde{T}) \to \mathbb{P}_p(\tilde{T}) \) the \( L^2(\tilde{T}) \)-projection. We will base our analysis on the univariate and \( C^1 \)-conforming \( hp \)-projectors \( \tilde{\pi}_{p,2} \) constructed in Ref. 5 (Section 5).

**Lemma 4.1.** For \( p \geq 3 \), there is a unique projector \( \tilde{\pi}_{p,2} : H^2(\tilde{T}) \to \mathbb{P}_p(\tilde{T}) \) that satisfies \( (\tilde{\pi}_{p,2}v)^{(2)} = \tilde{\pi}_{p-2,0}(v^{(2)}) \) and \( (\tilde{\pi}_{p,k}v)^{(j)}(\pm 1) = v^{(j)}(\pm 1) \) for \( j = 0, 1 \).

The projector \( \tilde{\pi}_{p,2} \) is stable in \( H^2(\tilde{T}) \), cp. Ref. 5 (Proposition 8.4). Moreover, \( hp \)-version approximation properties of \( \pi_{p,2} \) were established in Ref. 5 (Theorem 8.3).

Next, we tensorize the one-dimensional projectors \( \tilde{\pi}_{p,2} \) along the lines of Ref. 15 (Section 5.1.2). Let \( \tilde{T}^d = \tilde{T} \times \cdots \times \tilde{T} \) for \( d \geq 2 \). Coordinates in \( \tilde{T}^d \) are written as \( \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_d) \). We introduce the tensor-product Sobolev space \( H^2_{\text{mix}}(\tilde{T}^d) := \bigotimes_{i=1}^d H^2(\tilde{T}) \). Notice that, for \( d = 3 \), we have the inclusion

\[
H^6(\tilde{T}^3) \subset H^2_{\text{mix}}(\tilde{T}^3). \tag{4.1}
\]

We set \( \mathbb{Q}_p(\tilde{T}^d) := \bigotimes_{i=1}^d \mathbb{P}_p(\tilde{T}) \). On \( \tilde{T}^d \) and for \( p \geq 3 \), we now define the tensorized interpolation operator

\[
\tilde{\pi}_{p,2}^d = \bigotimes_{i=1}^d \tilde{\pi}_{p,2}^{(i)}, \tag{4.2}
\]

where \( \tilde{\pi}_{p,2}^{(i)} \) denotes the univariate projector defined in Lemma 4.1, acting in the variable \( \tilde{x}_i \). The projector is well-defined and stable on \( H^2_{\text{mix}}(\tilde{T}^d) \); see Ref. 15 (Proposition 5.3). For corresponding \( hp \)-approximation results, we refer to Ref. 15 (Proposition 5.4).

In the discussion of interelement continuity of \( \pi_{k,2}^d \), a crucial role is taken by the following property. It implies that traces of the tensorized interpolant and tensor projection commute. It is an immediate consequence of (4.2) and of the defining properties of the univariate projector in Lemma 4.1.

**Proposition 4.1.** For \( d \geq 2 \) and \( 1 \leq j \leq d \), there holds

\[
(\tilde{\pi}_{p,2}^d v)|_{\tilde{x}_j=\pm 1} = \left( \bigotimes_{1 \leq i \neq j \leq d} \tilde{\pi}_{p,2}^{(i)} \right) (v(\cdot, \tilde{x}_j = \pm 1)). \tag{4.3}
\]

**Remark 4.1.** Observe that (4.3) is recursive: one may take repeated traces with respect to a sequence \( \{\tilde{x}_j(k)\}_{k \geq 1} \) of coordinates, with \( j(k) \neq j(k') \) for \( k' \neq k \). In particular, by taking the traces twice, we see that \( (\tilde{\pi}_{p,2}^3 v)|_{\tilde{x}_i=\pm 1, \tilde{x}_j=\pm 1} \) corresponds to the univariate projections \( \tilde{\pi}_{p,2} \) of traces of \( v \) onto polynomials on all edges \( \{\tilde{x}_i = \pm 1\} \cap \{\tilde{x}_j = \pm 1\} \), for \( i \neq j \). Iterating this argument \( d \) times, we obtain nodal exactness of \( \tilde{\pi}_{p,2}^d v \) in the vertices of \( \tilde{T}^d \) follows: for \( v \in H^2_{\text{mix}}(\tilde{T}^d) \subset C^{0}(\tilde{T}^d) \), we have

\[
v(Q) = (\tilde{\pi}_{p,2}^d v)(Q) \quad \text{for all vertices } Q \text{ of } \tilde{T}^d. \tag{4.4}
\]
4.2. Continuity properties

For an axiparallel hexahedron $K$ and for a function $v : K \to \mathbb{R}$ with $\tilde{v} = v \circ \Phi_K \in H^2_{\text{mix}}(\tilde{K})$, we now define the elemental interpolant $(\pi^3_{p,2} v)|_K \in \mathbb{Q}_p(K)$ in a standard way by setting

$$(\pi^3_{p,2} v)|_K \circ \Phi_K := \tilde{\pi}^3_{p,2} (v \circ \Phi_K),$$

(4.5)

with $\tilde{\pi}^3_{p,2}$ the reference tensor interpolant (4.2) (for $d = 3$) and $\Phi_K : \tilde{K} \to K$ the elemental mapping. Analogously, we denote by $b$ the univariate projector $\tilde{\pi}_{p,2}$ applied in direction $x_i$ on element $K$.

The following lemma is a straightforward consequence of (4.4) and Remark 4.1, taking into account the inclusion (4.1).

Lemma 4.2. Let $K, K'$ be two axiparallel hexahedra, $F = F_{K,K'}$, a regularly matching face and $v \in H^6(\text{int}(\overline{K} \cup \overline{K'}))$. Then we have

$$(\pi^3_{p,2} v|_K)|_F = (\pi^3_{p,2} v|_{K'})|_F,$$

(4.6)

implying that the projector $\pi^3_{p,2}$ yields a piecewise polynomial approximation which is continuous across the regular face $F$.

Similarly, let $K$ be an axiparallel hexahedron, $F \in \mathcal{F}_K$ and $v \in H^6(K)$. If there holds $v|_F \equiv 0$, then we have

$$(\pi^3_{p,2} v|_K)|_F \equiv 0,$$

(4.7)

implying that the projector $\pi^3_{p,2}$ preserves homogeneous Dirichlet boundary conditions on the face $F$. Additionally, if $E \subset F$ is an elemental edge of $K$ (i.e., $E \in \mathcal{E}_K$), then identity (4.7) implies

$$(\pi^3_{p,2} v|_K)|_E \equiv 0.$$

(4.8)

4.3. Definition of the base projectors

Analogously to (3.4), we split the geometrically refined reference mesh patches into interior elements and terminal layer elements, with respect to the singular set on $\tilde{Q}$ induced via the patch maps by the corresponding singular corners and edges on $\Omega$:

$$\mathcal{M}^{\ell,t}_\sigma := \tilde{D}^{\ell,t}_\sigma \cup \tilde{\tau}^{\ell,t}_\sigma, \quad t \in \{c, e, ce\}.$$

(4.9)

For a function $\tilde{u} : \tilde{Q} \to \mathbb{R}$ and the polynomial degree $p_e$ in (3.11), we define the non-conforming reference base interpolant $\pi^{\ell,t}_b \tilde{u}$ elementwise as

$$\pi^{\ell,t}_b|_K \tilde{u} := \begin{cases} 
\pi_{p_e,2}|_K \tilde{u}|_K & t = \text{int and } \tilde{K} \in \mathcal{M}^{\ell,t}_\sigma \text{ int}, \\
\pi_{p_e,2}|_K \tilde{u}|_K & t \in \{c, e, ce\} \text{ and } \tilde{K} \in \mathcal{D}^{\ell,t}_\sigma, \\
0 & t \in \{c, e, ce\} \text{ and } \tilde{K} \in \mathcal{\tilde{\tau}}^{\ell,t}_\sigma.
\end{cases}$$

(4.10)

This interpolant is equal to the tensorized interpolant $\pi^3_{p,2}|_K \tilde{u}$ in (4.5) on all axiparallel hexahedral elements whose distance to the singular support of $\tilde{u}$ in $\overline{Q}$ is
positive, and equal to zero on all terminal layer elements (in the respective reference patches). Due to the inclusion (4.1), the base interpolants are well-defined for functions in the Sobolev space $M^s_t(Q)$ with weighting according to the patch type $t \in \{e, ce, int\}$.

**Remark 4.2.** In view of Lemma 4.2, the base interpolants $\tilde{\pi}^{\ell, t}_b \bar{u}$ in (4.10) are conforming over regularly matching faces within the reference mesh patch $\tilde{M}^t$ and satisfy homogeneous boundary conditions on patch faces corresponding to boundary faces of a geometric mesh $M$ on $\Omega$. However, $\tilde{\pi}^{\ell, t}_b \bar{u}$ is generally discontinuous over irregular faces within a reference mesh patch, as well as at the boundary of the terminal layers.

### 4.4. Exponential convergence on reference mesh patches

Next, we review from Ref. 15 the patchwise exponential convergence results in (3.24) for the reference base projectors in (4.10) and for solutions $u$ with patchwise analytic regularity for the pull-back $\tilde{u}$, i.e., $u \in A_k(Q)$ for $t \in \{e, ce, int\}$.

First, **interior patches** $M^e_p = G_p(M^e)$ consist of a fixed regular collection of axiparallel, hexahedral elements whose distances to the singular set $S$ are bounded away from zero. Hence, the analytic regularity reference class $A^{e}_k(Q)$ associated with $M^e_p$ consists of functions $u$ which are analytic in $\tilde{Q}$ with

$$|\tilde{u}|^2_{A^{e}_k(Q)} := |\tilde{u}|^2_{H^k(Q)} \leq C 2^{2(k+1)} \Gamma(k+1)^2, \quad k \in \mathbb{N}_0. \quad (4.11)$$

By proceeding as in Ref. 15 (Proposition 5.10), we obtain the following exponential bound.

**Proposition 4.2.** Let $\tilde{u}$ satisfy the regularity assumption (4.11). Consider the base projector $\tilde{\pi}^{\ell, e}_b \bar{u}$ in (4.10) with polynomial degrees $p_\ell \simeq \ell$ as in (3.11). Then, as $\ell \to \infty$, we have the error bound

$$\Upsilon_{\tilde{M}^e_{\ell, b} \bar{u}} \left[ \tilde{u} - \tilde{\pi}^{\ell, e}_b \bar{u} \right] \leq C \exp(-2b\ell),$$

with constants $b, C > 0$ independent of $\ell$.

Second, for the **reference corner mesh patch** $\tilde{M}^{ce}_\sigma$ on $\tilde{Q}$, the analytic reference class $A^{ce}_k(Q)$ can be assumed to consist of functions $\tilde{u}$ that satisfy

$$|\tilde{u}|^2_{\tilde{M}^{ce}_k(Q)} := \sum_{|\alpha|=k} ||r^{-1-b_e+|\alpha|}\tilde{D}^\alpha \tilde{u}||^2_{L^2(Q)} \leq C 2^{2(k+1)} \Gamma(k+1)^2, \quad k \in \mathbb{N}_0, \quad (4.12)$$

where $r_e$ is the distance to a corner of $\tilde{Q}$, cp. Figure 1 (left), and $b_e \in (0, 1)$ a corner weight exponent as in (2.12).

**Proposition 4.3.** Let $\tilde{u}$ satisfy the regularity assumption (4.12). Consider the base projector $\tilde{\pi}^{\ell, ce}_b \bar{u}$ in (4.10) with polynomial degrees $p_\ell \simeq \ell$ as in (3.11). Then, as $\ell \to \infty$, we have the error bound

$$\Upsilon_{\tilde{M}^{ce}_{\ell, b} \bar{u}} \left[ \tilde{u} - \tilde{\pi}^{\ell, ce}_b \bar{u} \right] \leq C \exp(-2b\ell),$$
with constants $b, C > 0$ independent of $\ell$.

**Proof.** From (4.9), we have $\widehat{M}_{\sigma}^{\ell, c} = \widehat{\Sigma}_{\sigma}^{\ell, c} \cup \widehat{\Sigma}_{\sigma}^{\ell, c}$. On $\widehat{\Sigma}_{\sigma}^{\ell, c}$, exponential convergence follows from Ref. 15 (Proposition 5.13). On the mesh $\widehat{\Sigma}_{\sigma}^{\ell, c}$, the exponential bound results from Ref. 15 (Proposition 5.21).

Third, for the reference edge mesh patch $\widehat{M}_{\sigma}^{\ell, e}$ on $\widehat{Q}$, we analogously shall consider the analytic reference class $A_{e}(\widehat{Q})$ of functions $\widehat{u}$ so that

$$|\widehat{u}|_{M_{k}^{e}(\widehat{Q})}^{2} := \sum_{|\alpha|=k} \left| r_{e}^{-1-\rho_{e}+|\alpha|} \Delta^{\alpha} c_{e} \right|_{L^{2}(\widehat{Q})}^{2} \leq C^{2(k+1)} \Gamma(k+1)^{2}, \quad k \in \mathbb{N}_{0},$$

(4.13)

where $r_{e}$ is the distance to an edge of $\widehat{Q}$, as indicated in boldface in Figure 1 (middle), and $b_{e} \in (0,1)$ an edge weight exponent as in (2.12).

**Proposition 4.4.** Let $\widehat{u}$ satisfy the regularity assumption (4.13). Consider the base projector $\widehat{u}_{e}^{\ell, e} \widehat{u}$ in (4.10) with polynomial degrees $p_{e} \simeq \ell$ as in (3.11). Then, as $\ell \to \infty$, we have the error bound

$$\Upsilon_{\widehat{M}_{e}^{\ell, e}}[\widehat{u} - \widehat{u}_{e}^{\ell, e}] \leq C \exp(-2b\ell),$$

with constants $b, C > 0$ independent of $\ell$.

**Proof.** This is a consequence of Ref. 15 (Proposition 5.15) and Ref. 15 (Proposition 5.22).

Finally, we consider the reference corner-edge mesh patch $\widehat{M}_{\sigma}^{\ell, ce}$ on $\widehat{Q}$. By superposition as in Ref. 15, we may restrict ourselves to the case of single corner $c$ with a single edge $e \in E_{c}$. The associated analytic reference class $A_{ce}(\widehat{Q})$ now consists of functions $\widehat{u}$ satisfying

$$|\widehat{u}|_{M_{k}^{e}(\widehat{Q})}^{2} := \sum_{|\alpha|=k} \left| r_{e}^{-1-\rho_{e}+|\alpha|} \Delta^{\alpha} c_{e} \right|_{L^{2}(\widehat{Q})}^{2} \leq C^{2(k+1)} \Gamma(k+1)^{2},$$

(4.14)

for any $k \in \mathbb{N}_{0}$, where $r_{c}$ and $\rho_{ee}$ are the distances to the corner-edge pair on $\widehat{Q}$ formed by $c$ and $e$, as indicated in Figure 1 (right). The weight exponents $b_{c}, b_{e} \in (0,1)$ are as in (2.12).

**Proposition 4.5.** Let $\widehat{u}$ satisfy the regularity assumption (4.14). Consider the base projector $\widehat{u}_{e}^{\ell, ce} \widehat{u}$ in (4.10) with polynomial degrees $p_{e} \simeq \ell$ as in (3.11). Then, as $\ell \to \infty$, we have the error bound

$$\Upsilon_{\widehat{M}_{e}^{\ell, ce}}[\widehat{u} - \widehat{u}_{e}^{\ell, ce}] \leq C \exp(-2b\ell),$$

with constants $b, C > 0$ independent of $\ell$.

**Proof.** This bound follows from Ref. 15 (Proposition 5.17) and Ref. 15 (Proposition 5.23).
5. Polynomial Trace Liftings

In this section, we construct the $H^1$-conforming reference patch projectors $\bar{\Pi}^{t,1} = \bar{\pi}^{t,1} + \bar{\mathcal{L}}_t$ in (3.16), (3.25), by modifying the base projectors $\pi^{t,1}_b$ with the introduction of suitable polynomial trace liftings $\bar{\mathcal{L}}_t$ over irregular edges and/or irregular faces. We then prove the bound (3.26) in Proposition 3.1 separately for each reference mesh patch. To that end, we simply write $\eta := \bar{\pi}^{t,1} u$ for the base $\text{hp}$-projector in (4.10) on the respective reference mesh patch $\mathcal{M}^{t,1}_\sigma$, $t \in \{c,e,ce,\text{int}\}$, and $\Pi := \bar{\Pi}^{t,1}$ for the resulting patch projector. When clear from the context, we will omit "tildas" to denote quantities on reference patches, and use the notation

\begin{equation}
\eta = u - \Pi u, \quad \eta_b = u - \pi_b u,
\end{equation}

for a generic patch function $u$ in the weighted space $u \in M^{t,1}_\sigma(\bar{Q})$ for $t \in \{c,e,ce,\text{int}\}$; cp. inclusion (4.1).

5.1. Interior patch $\tilde{\mathcal{M}}^{t,\text{int}}_\sigma$

For the interior reference mesh patch $\tilde{\mathcal{M}}^{t,\text{int}}_\sigma$, we take $\eta := \eta_b$ on $\tilde{\mathcal{M}}^{t,\text{int}}_\sigma$ (5.2).

Since $\tilde{\mathcal{M}}^{t,\text{int}}_\sigma$ contains finitely many regular refinements, $\Pi u$ in (5.2) yields a conforming piecewise polynomial approximation over the patch $\tilde{\mathcal{M}}^{t,\text{int}}_\sigma$, cp. Lemma 4.2 and Assumptions 3.1, 3.2. No further liftings are required, and the bound (3.26) hold trivially for $\tilde{\mathcal{M}}^{t,\text{int}}_\sigma$, without any loss in $p$.

Proposition 5.1. With (5.1) we have $\mathcal{T}_{\tilde{\mathcal{M}}^{t,\text{int}}_\sigma}[\eta] \lesssim \mathcal{T}_{\tilde{\mathcal{M}}^{t,\text{int}}_\sigma}[\eta_b]$.

5.2. Edge patch $\tilde{\mathcal{M}}^{t,e}_\sigma$

Next, we analyze the reference edge patch $\tilde{\mathcal{M}}^{t,e}_\sigma$, where irregular faces arise across layers due to irregular geometric refinements perpendicular to edges, as illustrated in Figure 1 (middle). As the edge-patch analysis will be a building block also for the corner-edge case, we consider here a more general edge patch $\tilde{\mathcal{M}}^{t,e}_\sigma$ whose edge-parallel size is determined by the parameter $h^\parallel$ (not necessarily of order one). Our estimates will then be made explicit in $h^\parallel$. According to (4.9), we write $\tilde{\mathcal{M}}^{t,e}_\sigma = \tilde{\mathcal{D}}^{t,e}_\sigma \cup \tilde{\mathcal{F}}^{t,e}_\sigma$ and consider the two submeshes separately.

5.2.1. Interior elements in $\tilde{\mathcal{D}}^{t,e}_\sigma$

By (3.5), the interior mesh $\tilde{\mathcal{D}}^{t,e}_\sigma$ can be partitioned into $\ell$ layers as $\tilde{\mathcal{D}}^{t,e}_\sigma = \bigcup_{\ell' = 0}^{\ell - 1} \tilde{\mathcal{D}}^{t,e}_{\ell'}$. By Lemma 4.2 and Remark 4.2, the base projector $\pi_b u$ is continuous over elements within each layer $\ell'$, and satisfies the homogeneous boundary conditions on appropriate patch boundary faces. For $1 \leq \ell' \leq \ell - 1$, we thus need to introduce trace liftings on the interface of the two adjacent mesh layers $\mathcal{L}^{t,e}_{\ell' - 1} = \{K_1,K_2,K_3\}$ and $\mathcal{L}^{t,e}_{\ell'} = \{K_1',K_2',K_3'\}$ as illustrated in Figure 2.
In the coordinate system there, the singular edge $e$ on the patch corresponds to $e = \{x = (x^\perp, x^\parallel) : x^\perp = a^+_1, x^\parallel = b^+_1, x^\parallel \in e^\parallel\}$, with $e^\parallel := (0, h^\parallel)$. As in (3.6), we write the elements $K_i, K'_i$ in the product form $K_i = K^\perp_i \times e^\parallel$ and $K'_i = (K^\perp_i)^\parallel \times e^\parallel$, respectively, where $K^\perp_i$ and $(K^\perp_i)^\parallel$ are shape-regular and axiparallel rectangles of diameters $\text{diam}(K^\perp_i) \simeq h^\perp$, with $h^\perp \lesssim h^\parallel$. We introduce the faces $f_{ij} := f_{K_i, K'_j}$, and note that $f_{21}, f_{22}, f_{31}, f_{33}$ are irregular; cp. Figure 2. We further set $f'_{ij} := f_{K'_i, K_j}$, and observe that $f'_{12}$ and $f'_{13}$ match regularly. As indicated in Figure 2, the precise locations of the elements, faces and edges in each layer are determined by the length parameters $a^+_1, a^+_2$ and $b^+_1, b^+_2$, which do not change from layer to layer. Finally, we note that $h^\parallel_{\text{min}, K_i} \simeq h^\parallel_{\text{min}, K'_j} \simeq h^\parallel$. We then consider the polynomial face jumps

$$
\left[\pi_b u\right]_{f_{ij}} = \left[\pi_b u\right]_{f_{K_i, K'_j}} = (\pi_b |_{K_i} u |_{K_i} - \pi_b |_{K'_j} u |_{K'_j})|_{f_{ij}}.
$$

Similarly, we denote the jumps across $f'_{ij}$ by $\left[\pi_b u\right]_{f'_{ij}} := \left[\pi_b u\right]_{f_{K'_i, K'_j}}.$

**Remark 5.1.** The jumps (5.3) have a natural tensor-product structure. To describe it, we write $f_{ij} := e^\perp_{ij} \times e^\parallel$. The tensor-product definitions (4.2), (4.5) imply the representation $\pi_b |_{K} = \pi_b^{(1)} |_{K} \otimes \pi_b^{(2)} |_{K}$, with $\pi_b^{(p)} |_{K} = \pi_p^{(1)} |_{K} \otimes \pi_p^{(2)} |_{K}$ the tensor-product projector acting in the first two coordinate directions and considered already in Ref. 16 (Theorem 4.72), and with $\pi_b^{(1)} |_{K}$ denoting the univariate projector $\pi_p^{(1)} |_{K}$ acting in edge-parallel direction. It can be readily seen that

$$
\left[\pi_b u\right]_{f_{ij}} = \left[\pi_b^1 u\right]_{e^\perp_{ij}} \otimes \pi_b^{(2)} |_{f_{ij}}, \quad \text{on } f_{ij} = e^\perp_{ij} \times e^\parallel,
$$

with $\left[\pi_b^1 u\right]_{e^\perp_{ij}}$ denoting two-dimensional jumps in direction perpendicular to the edge $e$ on the slice $x^\parallel \in e^\parallel$. (The jumps $\left[\pi_b u\right]_{f'_{ij}}$ have an analogous structure.)
We also point out that, for \( x^\parallel = 0 \) (and analogously for \( x^\parallel = h^\parallel \)), the nodal exactness property of the univariate projector \( \pi_b^\parallel \) in Lemma 4.1 implies
\[
(\pi_b u)(x^\perp, 0) = (\pi_b^0 u(\cdot, 0))(x^\perp), \quad x^\perp = (x^\parallel_1, x^\parallel_2) \in \{K^\parallel_i \cup (K^\parallel_i)^3\}_{i=1}^3.
\] (5.5)
In conjunction with Assumptions 3.1, 3.2, property (5.5) implies the conformity of the base projections \( \pi_u \) in edge-parallel direction over (the corresponding mesh layers of) different edge patches across \( x^\parallel = 0 \) (or \( x^\parallel = h^\parallel \)); see also the discussion in Remark 4.2.

The next lemma records several other results for the polynomial jumps (5.3). To state them, we introduce the elemental edges \( E_1 = \{x^\parallel_1 = 0, x^\parallel_2 = 0, x^\parallel = e^\parallel\} \), \( E_2 = \{x^\parallel_1 = a^\parallel_2, x^\parallel_2 = 0, x^\parallel = e^\parallel\} \) and \( E_3 = \{x^\parallel_1 = 0, x^\parallel_2 = b^\parallel_2, x^\parallel = e^\parallel\} \) as depicted in Figure 2.

**Lemma 5.1.** The jumps \([\pi_b u]_{f_{11}}, [\pi_b u]_{f_{22}}\) are continuous across \( x^\parallel_1 = a^\parallel_1 \), while the jumps \([\|\pi_b u\|_{f_{11}}, [\pi_b u]_{f_{11}}\) are continuous across \( x^\parallel_2 = b^\parallel_1 \). In addition, for the elemental edges \( E_1, E_2, E_3 \) in Figure 2, there holds
\[
[\pi_b u]_{f_{11}} \equiv 0 \text{ on } E_1, \quad [\|\pi_b u\|_{f_{11}}, [\pi_b u]_{f_{11}} \equiv 0 \text{ on } E_2, \quad [\pi_b u]_{f_{22}} \equiv 0 \text{ on } E_3. \quad (5.6)
\]
\[
[\pi_b u]_{f_{22}} \equiv 0 \text{ on } E_2, \quad [\|\pi_b u\|_{f_{22}}, [\pi_b u]_{f_{22}} \equiv 0 \text{ on } E_3. \quad (5.7)
\]

**Proof.** Let \( x = (a^\parallel_1, 0, x^\parallel) \). By (5.3) and since \( f_{21} \) and \( f_{22} \) lie on \( x^\parallel_2 = 0 \), we conclude that \([\pi_b u]_{f_{21}}(x) = (\pi_b|_{K^\parallel_2 \cup K^\parallel_2} - \pi_b|_{K^\parallel_1 \cup K^\parallel_1})(x) \), and \([\pi_b u]_{f_{22}}(x) = (\pi_b|_{K^\parallel_2 \cup K^\parallel_2} - \pi_b|_{K^\parallel_1 \cup K^\parallel_1})(x) \). Since \( K^\parallel_1 \) and \( K^\parallel_2 \) match regularly over \( f_{12} \), property (4.6) in Lemma 4.2 implies \( \pi_b|_{K^\parallel_1 \cup K^\parallel_1}(x) = \pi_b|_{K^\parallel_2 \cup K^\parallel_2}(x) \), which shows the continuity of \([\pi_b u]_{f_{21}} \) and \([\pi_b u]_{f_{22}} \) at \( x^\parallel_1 = a^\parallel_1 \). The continuity of \([\pi_b u]_{f_{11}} \) and \([\pi_b u]_{f_{22}} \) at \( x^\parallel_2 = b^\parallel_1 \) is proved in an analogous manner.

To establish (5.6), (5.7), we consider exemplarily the edge \( E_1 \subset f_{21} \) (the proof of the other cases is analogous). By using the tensor-structure (5.4) and the nodal exactness property (4.4) (for the perpendicular tensor projector \( \pi_b^\parallel \)), we see that, for \( x = (0, 0, x^\parallel) \), we have \([\pi_b u]_{f_{21}}(x) = (\Pi_b\Pi_{e^\parallel} \otimes \pi_b^\parallel)(x) = [\pi_b^\parallel u]_{f_{21}}(x) = 0 \), which finishes the proof.

The structure (5.4) of the polynomial jumps (5.3) motivates the introduction of the following lifting operator:
\[
\mathcal{L}_e[\pi_b u] := \begin{cases}
[\pi_b u]_{f_{21}}(1 - x^\parallel_2/b^\parallel_2) + [\pi_b u]_{f_{21}}(1 - x^\parallel_1/a^\parallel_1) & \text{on } K^\parallel_1,
[\pi_b u]_{f_{22}}(1 - x^\parallel_2/b^\parallel_2) & \text{on } K^\parallel_2,
[\pi_b u]_{f_{33}}(1 - x^\parallel_1/a^\parallel_1) & \text{on } K^\parallel_3,
0 & \text{on } \{K^\parallel_i\}_{i=1}^3.
\end{cases}
\] (5.8)

Obviously, \( \mathcal{L}_e[\pi_b u]|_{K^\parallel_i} \in Q_p(K^\parallel_i) \) for \( i = 1, 2, 3 \).

**Lemma 5.2.** The lifting \( \mathcal{L}_e[\pi_b u] \) belongs to \( C^0(K^\parallel_1 \cup K^\parallel_2 \cup K^\parallel_3) \). It vanishes on the sets \( \partial K^\perp_2 \cap \{x^\parallel_1 = a^\parallel_2\} \), \( \partial K^\perp_2 \cap \{x^\parallel_2 = b^\parallel_1\} \), \( \partial K^\perp_3 \cap \{x^\parallel_1 = a^\parallel_1\} \) and \( \partial K^\perp_3 \cap \{x^\parallel_2 = b^\parallel_2\} \),
implying that its support does not extend into layer $\ell' + 1$ and beyond the patch borders within layer $\ell'$. Moreover, we have

\begin{align}
(\mathcal{L}_e[\pi_b u]|_{\partial K_1^i})|_{f_{21}} &= [\pi_b u]_{f_{21}}, & (\mathcal{L}_e[\pi_b u]|_{\partial K_1^j})|_{f_{31}} &= [\pi_b u]_{f_{31}}, \quad (5.9) \\
(\mathcal{L}_e[\pi_b u]|_{\partial K_2^j})|_{f_{22}} &= [\pi_b u]_{f_{22}}, & (\mathcal{L}_e[\pi_b u]|_{\partial K_3^j})|_{f_{32}} &= [\pi_b u]_{f_{32}}, \quad (5.10)
\end{align}

**Proof.** The continuity of $\mathcal{L}_e[\pi_b u]$ over $\overline{K_1^i} \cup \overline{K_2^j} \cup \overline{K_3^j}$ follows readily from the construction of the lifting and Lemma 5.1.

From the definition (5.8) it follows further that the lifting vanishes on the sets $\partial K_2^j \cap \{x_2^+ = b_1^+\}$ and $\partial K_3^j \cap \{x_2^+ = a_3^+\}$. Next, we consider the boundary set $\partial K_1^j \cap \{x_1^+ = a_1^+\}$. By property (5.7) and since $f_{22}$ lies on $x_2^+ = 0$, there holds $[\pi_b u]|_{f_{22}}(a_3^+, 0, x^\parallel) = 0$. Hence, the lifting also vanishes on $\partial K_2^j \cap \{x_1^+ = a_1^+\}$. The proof for the set $\partial K_3^j \cap \{x_2^+ = b_2^+\}$ is analogous.

Finally, we verify the first identity in (5.9) for the face $f_{21} \subset \partial K_1^i$; the other identities are established analogously. There holds

$$
\mathcal{L}_e[\pi_b u]|_{\partial K_1^i}(x_1^+, 0, x^\parallel) = [\pi_b u]|_{f_{21}}(x_1^+, 0, x^\parallel) + [\pi_b u]|_{f_{31}}(0, 0, x^\parallel)(1 - x_1^+/a_1^+) .
$$

With (5.6), we have $[\pi_b u]|_{f_{31}}(0, 0, x^\parallel) = 0$. Hence, $(\mathcal{L}_e[\pi_b u]|_{\partial K_1^i})|_{f_{21}} = [\pi_b u]|_{f_{21}}$.

**Remark 5.2.** From (5.4), we have $\mathcal{L}_e[\pi_b u] = \mathcal{L}_e[\pi_b^* u] \otimes \pi_b^\parallel$, where $\mathcal{L}_e[\pi_b^*]$ is a (slight modification of) the two-dimensional polynomial trace lifting operator introduced in Ref. 16 (Theorem 4.72) in the slice at $x^\parallel \in e^\parallel$ in edge-perpendicular direction; cp. Figure 2. Hence, for $x^\parallel = 0$ (and analogously for $x^\parallel = h^\parallel$), property (5.5) implies

$$
\mathcal{L}_e[\pi_b u](x^\perp, 0) = (\mathcal{L}_e[\pi_b^* u](\cdot, 0)](x^\perp) , \quad x^\perp \in (K_1^i)^\perp , \quad i = 1, 2, 3 . \quad (5.11)
$$

The lifting $\mathcal{L}_e[\pi_b u]$ does not generally vanish at the edge-perpendicular patch borders situated on $x^\parallel = 0$ or $x^\parallel = h^\parallel$. However, under Assumptions 3.1, 3.2 and upon introducing corresponding liftings in the adjacent edge or corner-edge patches, the tensor-product structure of the base projectors and liftings in (5.5) and (5.11), respectively, guarantee the conformity of the patch projectors $\mathcal{L}_e[\pi_b u]$ over the corresponding mesh layers of different edge patches on the edge-perpendicular patch interfaces $x^\parallel = 0$ or $x^\parallel = h^\parallel$.

Lemma 5.2 and the definition of the jumps (5.3) imply that the lifted projector $\Pi u := \pi_b u + \mathcal{L}_e[\pi_b u]$ gives a continuous and piecewise polynomial approximation over mesh layers $\ell' - 1$ and $\ell'$, which does not affect the base projection $\pi_b u$ at the interface to the next layer $\ell' + 1$. In view of Lemma 5.2 and property (4.7), $\Pi u$ satisfies homogeneous Dirichlet boundary conditions which might possibly arise within layer $\ell'$ on $\partial K_2^j \cap \{x_1^+ = a_1^+\}$ or $\partial K_3^j \cap \{x_2^+ = b_2^+\}$.

As we show next, the jump lifting $\mathcal{L}_e[\pi_b u]$ is stable and the (weighted) $H^1$-norm of $\eta = u - \Pi u$ can be controlled in terms of $\eta_b = u - \pi_b u$. 
Lemma 5.3. We have the bounds

\[ N_{K'_1}[\mathcal{L}_e[\pi_b u]]^2 \lesssim p^4(h^\perp)^{-1}\left(\|\pi_b u\|_{L^2(f_{21})}^2 + \|\pi_b u\|_{L^2(f_{31})}^2\right), \]

\[ N_{K'_2}[\mathcal{L}_e[\pi_b u]]^2 \lesssim p^4(h^\perp)^{-1}\left(\|\pi_b u\|_{L^2(f_{22})}^2\right), \]

\[ N_{K'_3}[\mathcal{L}_e[\pi_b u]]^2 \lesssim p^4(h^\perp)^{-1}\left(\|\pi_b u\|_{L^2(f_{32})}^2\right), \]

and

\[ \sum_{i=1}^3 (N_{K'_i}[^{\eta}]^2 + N_{K'_i}[^{\eta}b]^2) \lesssim p^4 \sum_{i=1}^3 (N_{K'_i}[^{\eta}]^2 + N_{K'_i}[^{\eta}b]^2). \]

Proof. We only prove (5.12) for \( K'_1 \) (the other bounds are analogous). To do so, we first assume that the configuration is of unit size, i.e., \( h^\perp \simeq h^\parallel \simeq O(1) \). Then, by definition of \( \mathcal{L}_e[\pi_b u] \) in (5.8), it readily follows that

\[ \|\mathcal{L}_e[\pi_b u]\|_{L^2(K'_1)}^2 \lesssim (\|\pi_b u\|_{L^2(f_{21})}^2 + \|\pi_b u\|_{L^2(f_{31})}^2). \]

From the inverse inequality in Ref. 16 (Theorem 3.91), we find that

\[ \|\nabla v\|_{L^2(\tilde{K})}^2 \lesssim p^4\|v\|_{L^2(\tilde{K})}^2 \quad \forall \ v \in \mathcal{Q}_p(\tilde{K}). \]

Hence,

\[ \|\nabla \mathcal{L}_e[\pi_b u]\|_{L^2(K'_1)}^2 \lesssim p^4(\|\pi_b u\|_{L^2(f_{21})}^2 + \|\pi_b u\|_{L^2(f_{31})}^2). \]

In the general case as shown in Figure 2, we use the bounds above in conjunction with the anisotropic scalings in Lemma 3.1, exploiting the assumption \( h^\perp \lesssim h^\parallel \lesssim 1 \). This results in

\[ \|\mathcal{L}_e[\pi_b u]\|_{L^2(K'_1)}^2 \lesssim h^\perp(\|\pi_b u\|_{L^2(f_{21})}^2 + \|\pi_b u\|_{L^2(f_{31})}^2), \]

\[ \|\nabla \mathcal{L}_e[\pi_b u]\|_{L^2(K'_1)}^2 \lesssim p^4(h^\perp)^{-1}(\|\pi_b u\|_{L^2(f_{21})}^2 + \|\pi_b u\|_{L^2(f_{31})}^2). \]

From these bounds and since \( h^\perp_{\text{min},K'_1} \simeq h^\perp \), we conclude that

\[ N_{K'_1}[\mathcal{L}_e[\pi_b u]]^2 \lesssim p^4(h^\perp)^{-1}(\|\pi_b u\|_{L^2(f_{21})}^2 + \|\pi_b u\|_{L^2(f_{31})}^2), \]

which proves the bound (5.12) for \( K'_1 \).

To establish (5.13), we note that, with the triangle inequality,

\[ \sum_{i=1}^3 (N_{K'_i}[\eta]^2 + N_{K'_i}[\eta b]^2) \lesssim \sum_{i=1}^3 (N_{K'_i}[\eta]^2 + N_{K'_i}[\eta b]^2 + N_{K'_i}[\mathcal{L}_e[\pi_b u]]^2). \]

We next estimate the term \( N_{K'_2}[\mathcal{L}_e[\pi_b u]]^2 \). From (5.12) and (3.9) (with the fact that \( \|\pi_b u\|_{f_2} = [\eta b]_{f_2} \)), we see that

\[ N_{K'_2}[\mathcal{L}_e[\pi_b u]]^2 \leq p^4(h^\perp)^{-1}\|\pi_b u\|_{L^2(f_{22})}^2 \lesssim p^4\left( N_{K'_2}[\eta]^2 + N_{K'_2}[\eta b]^2 \right). \]

Similar bounds for \( N_{K'_1}[\mathcal{L}_e[\pi_b u]] \) and \( N_{K'_3}[\mathcal{L}_e[\pi_b u]] \) yield the assertion. \( \square \)
5.2.2. Terminal layer elements in $\tilde{\Gamma}^{\ell,e}_\sigma$

We next consider the interface from $\tilde{\Omega}^{\ell,e}_\sigma$ to $\tilde{\Gamma}^{\ell,e}_\sigma$ illustrated in Figure 3. We use the same notations as in Section 5.2.1. The set $\tilde{\Omega}^{\ell-1,e}_\sigma = \{K_1, K_2, K_3\}$ forms layer $\ell - 1$ of $\tilde{\Omega}^{\ell,e}_\sigma$, and $\tilde{\Gamma}^{\ell,e}_\sigma = \{K'_1\}$ is the terminal layer. The base projector $\pi_b u$ is set to zero in $K'_1$ as per (4.10). Thus, even though the faces $f_{21}$ and $f_{31}$ are now regularly matching, the jumps $[\pi_b u]_{f_{21}}$ and $[\pi_b u]_{f_{31}}$ do not vanish in general. In addition, the key properties (5.6), (5.7) are not valid anymore. On the other hand, due to Assumptions 3.1, 3.2, the base projectors $\pi_b u$ are continuous in parallel direction over (the corresponding mesh layers of) different edge patches across $x^b = 0$ or $x^e = h^e$; cp. Remark 5.1 and the definition of the base projectors in (4.10). To overcome the above-mentioned difficulties, we modify the lifting procedure in (5.8) by introducing suitable edge liftings associated with the edges $E_1, E_2, E_3$ shown in Figure 3. We detail this for edge $E_1$.

**Lemma 5.4.** We have $[\pi_b u]_{f_{21}} = [\pi_b u]_{f_{31}}$ on $\tilde{E}_1$. Analogous identities hold on $E_2$ and $E_3$ (across patch borders).

**Proof.** Let $x \in \tilde{E}_1$. Then, since $\pi_b u = 0$ on $K'_1$, we have $[\pi_b u]_{f_{21}}(x) = \pi_b|_{K_2} u|_{K_2}(x)$ and $[\pi_b u]_{f_{31}}(x) = \pi_b|_{K_3} u|_{K_3}(x)$. Since the elements $K_1, K_2, K_3$ in layer $\ell - 1$ match regularly, property (4.6) implies $\pi_b|_{K_2} u|_{K_2}(x) = \pi_b|_{K_3} u|_{K_3}(x)$, which proves the assertion. \[\square\]

With Lemma 5.4, we set $[\pi_b u]_{E_1} := [\pi_b u]_{f_{21}}|_{E_1} = [\pi_b u]_{f_{31}}|_{E_1}$, and introduce
the edge jump lifting
\[ L^E_1[\pi_b u] := \begin{cases} 
[\pi_b u]_{E_1} (1 - x_1^+ / a_1^+) (1 - x_2^+ / b_1^+) & \text{on } K'_1, \\
0 & \text{on } \{ K_i \}_{i=1}^3.
\end{cases} \tag{5.17} \]

Clearly, \( L^E_1[\pi_b u] \in \mathcal{Q}_p(K'_1) \). Furthermore, the lifting vanishes on the elemental faces \( \partial K'_1 \cap \{ x_1^+ = a_1^+ \} \) and \( \partial K'_1 \cap \{ x_2^+ = b_1^+ \} \); cp. Figure 3.

**Remark 5.3.** The lifting \( L^E_1 \) has the tensor-product structure \( L^E_1 = L^E_N \otimes \pi_b \), where the lifting \( L^E_N \) is a two-dimensional nodal lifting in edge-perpendicular direction into \( (K'_1)^\perp \) on the slice \( x_\parallel \in e_\parallel \) with \( N = (0, 0, x_\parallel) \).

**Lemma 5.5.** We have the bounds
\[ N_{K'_1}[L^E_1[\pi_b u]]^2 \lesssim \rho^6 (h^+)^{-1} \|[\pi_b u]_{f_{21}}\|_{L^2(f_{21})}^2, \tag{5.18} \]
\[ N_{K'_1}[L^E_1[\pi_b u]]^2 \lesssim \rho^6 (h^+)^{-1} \|[\pi_b u]_{f_{31}}\|_{L^2(f_{31})}^2, \]
and
\[ N_{K'_1}[L^E_1[\pi_b u]]^2 \lesssim \rho^6 (N_{K_2}[\eta_b]^2 + N_{K'_1}[\eta_b]^2), \]
\[ N_{K'_1}[L^E_1[\pi_b u]]^2 \lesssim \rho^6 (N_{K_3}[\eta_b]^2 + N_{K'_1}[\eta_b]^2). \tag{5.19} \]

**Proof.** We establish the bounds associated with face \( f_{21} \); the proof of the other ones is analogous. If the configuration in Figure 3 is of unit size, the two-dimensional polynomial trace inequality in Ref. 16 (Theorem 4.76) yields
\[ \|[\pi_b u]_{E_1}\|_{L^2(E_1)}^2 \lesssim \rho^2 \|[\pi_b u]_{f_{21}}\|_{L^2(f_{21})}^2, \tag{5.20} \]
since \( [\pi_b u]_{E_1} = [\pi_b u]_{f_{21}}|_{E_1} \). With this estimate, it readily follows that
\[ \|L^E_1[\pi_b u]\|_{L^2(K'_1)}^2 \lesssim \|[\pi_b u]_{E_1}\|_{L^2(E_1)}^2 \lesssim \rho^2 \|[\pi_b u]_{f_{21}}\|_{L^2(f_{21})}^2. \]
As before, the inverse inequality (5.14) implies
\[ \|\nabla L^E_1[\pi_b u]\|_{L^2(K'_1)}^2 \lesssim \rho^6 \|[\pi_b u]_{f_{21}}\|_{L^2(f_{21})}^2. \]
This implies (5.18) in the reference case. In the general case, we apply the scalings in Lemma 3.1, cp. (5.15) and (5.16), and obtain the first bound in (5.18).

This bound, the jump estimate (3.9) (with the fact that \([u]_{f_{21}} = 0\)) and the triangle inequality yield
\[ N_{K'_1}[L^E_1[\pi_b u]]^2 \lesssim \rho^6 (N_{K_2}[\eta_b]^2 + N_{K'_1}[\eta_b]^2). \]
This shows the first estimate in (5.19).

Next, we define the full edge lifting
\[ L^E[\pi_b u] = \sum_{j=1}^3 L^E_j[\pi_b u], \tag{5.21} \]
where \( L^E_j[\pi_b u], L^E_j[\pi_b u] \) are edge liftings associated with the edges \( E_2, E_3 \) in Figure 3 and defined analogously to (5.17). If the elemental edge \( E_2 \) or \( E_3 \) corresponds
to a boundary edge, the resulting edge lifting is identically zero, in accordance with property (4.8) for element \(K_2\) or \(K_3\). From stability bounds analogous to (5.18), we conclude that

\[
N_{K_1} [L^E_e [\pi_b u]]^2 \lesssim p^6 (N_{K_2} [\eta_b]^2 + N_{K_3} [\eta_b]^2 + N_{K_1} [\eta_b]^2).
\]  

(5.22)

**Remark 5.4.** The introduction of corresponding edge liftings \(L^E_e [\pi_b u]\) in terminal layer elements of adjacent edge patches readily yields continuity across patch borders in \((x^\pm)\)-direction or \((x^\pm)\)-direction. In addition, the tensor-product property in Remark 5.3 ensures the conformity of borders in terminal layer elements across \(x^\parallel = 0\) or \(x^\parallel = h^\parallel\) over edge or corner-edge patches; cp. Assumptions 3.1, 3.2.

By construction, the projector \(\pi^E_b u\) satisfies properties (5.6), (5.7) in the terminal layer setting of Figure 3.

**Lemma 5.6.** There holds:

\[
[\pi^E_b u]_{f_{21}} \equiv 0 \text{ on } E_1, \quad [\pi^E_b u]_{f_{31}} \equiv 0 \text{ on } E_1,
\]

(5.24)

\[
[\pi^E_b u]_{f_{21}} \equiv 0 \text{ on } E_2, \quad [\pi^E_b u]_{f_{31}} \equiv 0 \text{ on } E_3.
\]

(5.25)

**Proof.** We verify (5.24) for the face \(f_{21}\) (the proof for the other faces is analogous). By construction and the definition (5.3) of the jumps, we have, for \(x \in \overline{E}_1\),

\[
[\pi^E_b u]_{f_{21}} (x) = [\pi_b u]_{f_{21}} (x) - (L^E_i [\pi_b u]|_{K_i})_{f_{21}} (x) = [\pi_b u]_{f_{21}} (x) - [\pi_b u]_{f_{21}} (x) = 0.
\]

This yields the assertion.

Next, we adapt the polynomial face jump lifting \(L_e [\cdot]\) in (5.8) to the configuration in Figure 3:

\[
L^E_{K_1} [\pi_b u] := \begin{cases} [\pi^E_b u]_{f_{21}} (1 - x^1_+/b^+_1) + [\pi^E_b u]_{f_{31}} (1 - x^1_+/a^+_1) & \text{on } K_1, \\ 0 & \text{on } \{K_i\}_{i=1}^3.
\end{cases}
\]

(5.26)

With (5.24), (5.25), the lifting \(L^E_{K_1} [\pi_b u]\) vanishes on \(\partial K_1 \cap \{x^1_+ = a^+_1\}\) and \(\partial K_1 \cap \{x^1_+ = b^+_1\}\); cp. Lemma 5.2. Furthermore, \(L^E_{K_1} [\pi_b u]\) has a tensor-product structure similar to (5.11).

**Lemma 5.7.** There holds

\[
N_{K_1} [L^E_{K_1} [\pi_b u]]^2 \lesssim p^{10} (N_{K_2} [\eta_b]^2 + N_{K_3} [\eta_b]^2 + N_{K_1} [\eta_b]^2).
\]

(5.27)

**Proof.** By proceeding as in the proof of (5.12), we see that

\[
N_{K_1} [L^E_{K_1} [\pi_b u]]^2 \lesssim p^4 (h^+)^{-1} (\| [\pi^E_b u]_{f_{21}} \|^2_{L^2(f_{21})} + \| [\pi^E_b u]_{f_{31}} \|^2_{L^2(f_{31})}).
\]
Then, the jump bound (3.9) (noting that \([u]_{f_{21}} = [u]_{f_{31}} = 0\), the definition of the edge lifting and the triangle inequality yield

\[
N_{K_1'} [L^K_{e}[\pi_b u]]^2 \lesssim p^4 \left( N_{K_2} [\eta_b]^2 + N_{K_3} [\eta_b]^2 + N_{K_1'} [u - \pi^F_b u]^2 \right)
\]

\[
\lesssim p^4 \left( N_{K_2} [\eta_b]^2 + N_{K_3} [\eta_b]^2 + N_{K_1'} [\pi^F_b u]^2 \right).
\]

Invoking (5.22) yields (5.27).

In the configuration of Figure 3, we finally introduce the following lifting:

\[
L^T_{e}[\pi_b u] := \begin{cases} 
L^E_{e}[\pi_b u] + L^{K_1'}_{e}[\pi_b u] & \text{on } K_1' \\
0 & \text{on } \{K_i\}_{i=1}^3.
\end{cases}
\]  

(5.28)

By Remark 5.4, the edge jump liftings \(L^E_{e}[\pi_b u]\) in (5.28) give rise to a conforming function across neighboring patches. The lifting \(L^{K_1'}_{e}[\pi_b u]\) is piecewise polynomial and vanishes at the patch borders \(?K_1' \cap \{x^1 = a^1_1\}\) and \(?K_1' \cap \{x^1 = b^1\}\). Hence, it preserves essential boundary data which possibly arise on these patch faces.

**Remark 5.5.** The lifting \(L^T_{e}[\pi_b u]\) does not generally vanish on \(x^\parallel = 0\) or \(x^\parallel = h^\parallel\).

However, as in Section 5.2.1, the tensor-product structure of the liftings \(L^E_{e}[\pi_b u]\) and \(L^{K_1'}_{e}[\pi_b u]\) ensures the continuity of \(L^T_{e}[\pi_b u]\) in terminal layer elements across \(x^\parallel = 0\) or \(x^\parallel = h^\parallel\); cp. Assumptions 3.1, 3.2 and the nodal exactness of the univariate projector \(\pi^C_{b\parallel}\) in edge-parallel direction in Lemma 4.1.

**Lemma 5.8.** There holds:

\[
(L^T_{e}[\pi_b u]|_{K_1'})_{f_{21}} = [\pi_b u]_{f_{21}}, \quad (L^T_{e}[\pi_b u]|_{K_1'})_{f_{31}} = [\pi_b u]_{f_{31}}.
\]  

(5.29)

**Proof.** We show (5.29) for \(f_{21}\) (the proof for \(f_{31}\) is analogous). By (5.24) and as in the proof of (5.9), we have \((L^{K_1'}_{e}[\pi_b u]|_{K_1'})_{f_{21}} = [\pi^F_b u]_{f_{21}}\). Hence, with the definition of \(\pi^F_b u\) and the jumps in (5.23) and (5.3), respectively, we conclude that

\[
(L^T_{e}[\pi_b u]|_{K_1'})_{f_{21}} = (L^E_{e}[\pi_b u]|_{K_1'})_{f_{21}} + [\pi_b u]_{f_{21}} - (L^E_{e}[\pi_b u]|_{K_1'})_{f_{21}} = [\pi_b u]_{f_{21}},
\]

which gives the assertion.

In view of Remark 5.5 and Lemma 5.8, the lifted projector \(\Pi u := \pi_b u + L^T_{e}[\pi_b u]\) yields a piecewise polynomial and conforming approximation in the setting of Figure 3. The analog of the bound (5.13) in Lemma 5.3 reads as follows.

**Lemma 5.9.** We have the bound

\[
\sum_{i=1}^3 N_{K_i} [\eta]^2 + N_{K_1'} [\eta]^2 \lesssim p^{10} \left( \sum_{i=1}^3 N_{K_i} [\eta_b]^2 + N_{K_1'} [\eta_b]^2 \right).
\]  

(5.30)
submeshes separately. We proceed as in the edge patch case, by writing

\[ F = \sum_{i=1}^{3} N_{K_i}[\eta_i]^2 + N_{K'_i}[\eta_i]^2 \]

\[ \lesssim \sum_{i=1}^{3} N_{K_i}[\eta_i]^2 + N_{K'_i}[\eta_i]^2 + N_{K_i}[L^E_{e}[\pi_{b}u]]^2 + N_{K'_i}[L^E_{e}[\pi_{b}u]]^2. \]

Referring to (5.22) and Lemma 5.7 completes the proof.

Lemma 5.3 (for submesh \( \mathcal{D}_e^\sigma \)) and Lemma 5.9 (for submesh \( \mathcal{T}_e^\sigma \)) show the bound (3.26) over the reference edge patch \( \mathcal{M}_e^\sigma \).

**Proposition 5.2.** With (5.1) we have \( \mathcal{T}_{\mathcal{M}_e^\sigma, \mathcal{M}_e^\sigma} \lesssim p^{10} \mathcal{T}_{\mathcal{M}_e^\sigma, \mathcal{M}_e^\sigma} \).

### 5.3. Corner patch \( \mathcal{M}_e^c \)

We consider the reference corner patch \( \mathcal{M}_e^c \), where irregular faces appear due to hanging nodes located in the interior of faces, cp. Figure 1 (left). We proceed as in Section 5.2, by writing \( \mathcal{M}_e^c = \mathcal{D}_e^c \cup \mathcal{T}_e^c \), cp. (4.9), and by analyzing the two submeshes separately.

#### 5.3.1. Interior elements in \( \mathcal{D}_e^c \)

As in the edge patch case, we partition \( \mathcal{D}_e^c \) into \( \ell \) layers: \( \mathcal{D}_e^c = \bigcup_{i=0}^{\ell-1} \mathcal{D}_e^c(i); \) cp. (3.5). For \( 1 \leq \ell' \leq \ell - 1 \), we consider the interface of the two adjacent mesh layers \( \mathcal{D}_e^{c-1} = \{K_i\}_{i=1}^7 \) and \( \mathcal{D}_e^{c} = \{K'_i\}_{i=1}^7 \) shown in Figure 4.

All elements involved are shape-regular (uniformly in \( \ell \) and we may assume

\[ h_{K_i} \simeq h_{K'_i} \simeq h_{\min,K_i} \simeq h_{\min,K'_i} \simeq h, \quad 1 \leq i \leq 9, \]

for a mesh size parameter \( h \). As in Section 5.2, we introduce the faces \( f_{ij} = f_{K_i,K'_j} \) and \( f'_{ij} = f'_{K'_i,K'_j} \). The faces \( f_{21}, f_{22}, f_{76}, f_{77} \) are indicated in Figure 4. The locations of the elements, faces and edges in this configuration are determined by the length parameters \( a_1, a_2 \) in \( x_1 \)-direction, by \( b_1, b_2 \) in \( x_2 \)-direction, and by \( c_1, c_2 \) in \( x_3 \)-direction, respectively. Again, these values do not change from layer to layer. By Remark 4.2, the base projector \( \pi_{b}u \) defined in (4.10) is continuous across elements \( K_i, K_j \) within layer \( \ell' - 1 \), respectively across the faces \( f_{ij} \) within layer \( \ell' \), and satisfies possible homogeneous Dirichlet boundary conditions on patch boundary faces. The polynomial face jumps \( [\pi_{b}u]_{f_{ij}} \) are defined as in (5.3).

As further illustrated in Figure 5, there are three faces which constitute the interface between layer \( \ell' - 1 \) and \( \ell' \): \( F_1 := \partial K_2 \cap \{ x_2 = 0 \} \), \( F_2 := \partial K_7 \cap \{ x_3 = 0 \} \), and \( F_3 := \partial K_5 \cap \{ x_1 = 0 \} \). The elemental edges of these faces are denoted by \( E_1, \ldots, E_9 \) and are also depicted in Figure 5.

We define the polynomial jump across face \( F_1 \) by

\[ [\pi_{b}u]_{F_1} := [\pi_{b}u]_{f_{2j}} \quad \text{on} \quad f_{2j}, \quad 1 \leq j \leq 4. \]
The jumps $\|\pi_6 u\|_{F_2}$ and $[\pi_6 u]_{F_3}$ are defined analogously.
Lemma 5.10. We have \( [\pi_1 u]_{E_1} \in C^0(\mathcal{T}_1) \) and
\[
[\pi_1 u]_{E_1}(0, 0, 0) = 0, \quad [\pi_1 u]_{E_1}(0, 0, c_2) = 0,
\]
\[
[\pi_1 u]_{E_1}(a_2, 0, 0) = 0, \quad [\pi_1 u]_{E_1}(a_2, 0, c_2) = 0.
\]
Moreover, \( [\pi_1 u]_{E_1} = [\pi_1 u]_{E_3} \) on \( E_1 \). (Analogous properties hold for the jumps \( [\pi_1 u]_{E_2} \) and \( [\pi_1 u]_{E_3} \), possibly across patch borders.)

**Proof.** The continuity of \( [\pi_1 u]_{E_1} \) follows from the fact that elements \( K_1, \ldots, K_4 \) match regularly across the faces \( f_{12}, f_{14}, f_{15}, f_{25} \). Properties (5.33), (5.34) are direct consequences of the nodal exactness (4.4).

For the second statement, we proceed as in the proof of Lemma 5.4 and consider \( x \in E_1 \cap \partial K_j \) for \( j \in \{1, 3\} \). Then,
\[
[\pi_1 u]_{E_1}(x) = [\pi_1 u]_{E_3}(x) = \pi_1|_{K_1}(x) - \pi_1|_{K_4}(x),
\]
\[
[\pi_1 u]_{E_1}(x) = [\pi_1 u]_{E_3}(x) = \pi_1|_{K_3}(x) - \pi_1|_{K_4}(x).
\]
Since the elements \( \{K_j\}_{j=1}^7 \) in layer \( l' - 1 \) match regularly, by property (4.6), we obtain \( \pi_1|_{K_1}(x) = \pi_1|_{K_3}(x) \). This completes the proof. \( \square \)

We begin our construction by introducing polynomial edge-jump liftings associated with edges \( E_1, \ldots, E_9 \) in Figure 5; cp. Section 5.2.2. We detail this for edge \( E_1 \), the treatment of the other edges being analogous. With Lemma 5.10, we may set \( [\pi_1 u]_{E_1} := ([\pi_1 u]_{F_1})_{E_1} = ([\pi_1 u]_{F_3})_{E_1} \), and define the edge lifting
\[
L^E_1[\pi_1 u] := \begin{cases} 
[\pi_1 u]_{E_1}(1 - x_1/a_1)(1 - x_2/b_1) & \text{in } K_1', K_3', \\
0 & \text{otherwise}.
\end{cases}
\]

The lifting \( L^E_1[\pi_1 u] \) defines a piecewise polynomial and continuous function in \( K_1' \cup K_3' \); cp. Lemma 5.10, which reproduces the edge jump \( [\pi_1 u]_{E_1} \) on \( E_1 \). By construction and due to (5.34) it vanishes on edges \( E_2, E_4 \), as well as on \( \{x_1 = a_1\}, \{x_2 = b_1\} \). By proceeding as in the proof of (5.19) (with isotropic element scaling), the following bound can be readily verified.

Lemma 5.11. We have the bound
\[
N_{K_1}[L^E_1[\pi_1 u]]^2 + N_{K_3}[L^E_1[\pi_1 u]]^2 \lesssim \rho^6 \left( N_{K_2}[\eta_b]^2 + N_{K_4}[\eta_b]^2 + N_{K_3}[\eta_b]^2 \right).
\]

In the setting of Figure 5, we then define the full edge lifting \( L^E_c[\pi_1 u] \) as
\[
L^E_c[\pi_1 u] := \sum_{j=1}^9 L^E_j[\pi_1 u],
\]
where \( L^E_j[\pi_1 u] \) is an edge lifting associated with \( E_j \) as in (5.35). The function \( L^E_c[\pi_1 u] \) is piecewise polynomial and continuous in \( K_1' \cup \ldots \cup K_7' \).

**Remark 5.6.** If \( E_j \) corresponds to a boundary edge, the resulting edge lifting is identically zero, cp. (4.8). As in Remark 5.4, the introduction of corresponding edge
jump liftings in adjacent patches yields conformity across the patch border into an adjacent corner or corner-edge patch; cp. Assumptions 3.1, 3.2.

From estimates as in (5.36), we conclude that, over face $F_1$, we have

$$
su_{i=1}^{4} N_{K_i'}[\mathcal{L}_E^{F_i}[\pi_b u]]^2 \lesssim p^6 (N_{K_2} |\eta_b|^2 + \sum_{i=1}^{4} N_{K_i'} |\eta_b|^2). \tag{5.38}
$$

Similar bounds hold over $F_2$ and $F_3$. Proceeding as in Section 5.2.2, we introduce

$$\pi_b^E u := \pi_b u + \mathcal{L}_E^{F_i}[\pi_b u]. \tag{5.39}
$$

**Lemma 5.12.** We have $\|\pi_b^E u\|_{F_1} \in C^0(\overline{F_1})$ and $\|\pi_b^E u\|_{F_1} = 0$ on $\overline{F}_j$ for $1 \leq j \leq 4$. (Analogous properties hold for $F_2$ and $F_3$.)

**Proof.** We note that $[\pi_b^E u]_{F_i} = [\pi_b u]_{f_{2j} - (\mathcal{L}_E^{F_i}[\pi_b u])}_{f_{2j}}$ on $f_{2j}$ for $1 \leq j \leq 4$. Hence, the continuity of $[\pi_b^E u]_{F_1}$ follows from the continuity of $[\pi_b u]_{F_1}$ shown in Lemma 5.10 and from the continuity of the edge jump liftings.

Next, we show that $[\pi_b^E u]_{F_1} = 0$ on $F_1$; the proof for the other edges is analogous. Let $x \in \overline{F}_1 \cap \partial K_j$ for $j \in \{1,3\}$. We find that $\|\pi_b^E u\|_{F_1}(x) = [\pi_b u]_{f_{2j}}(x) - [\pi_b u]_{f_{2j}}(x) = 0$.

We lift the polynomial face jump $[\pi_b^E u]_{F_1}$ over $F_1$ from $K_2$ into layer $\ell'$ by

$$\mathcal{L}_E^{F_1,K_2}[\pi_b u] := \begin{cases} [\pi_b^E u]_{F_1}(1 - x_2/b_1) & \text{on } \{K_1\}_{i=1}^4, \\
0 & \text{otherwise}. \end{cases} \tag{5.40}
$$

Notice that, due to Lemma 5.12, this lifting vanishes on the sets $\{x : (x_1, 0, x_3) \in \partial F_1, x_2 \in (0, b_1)\}$ and $\{x_2 = b_1\}$, respectively, and in particular on the other faces $F_2$ and $F_3$. In addition, it reproduces the jump of $\pi_b^E u$ on $F_1$:

$$[\mathcal{L}_E^{F_1,K_2}[\pi_b u]|_{K_j' \cap F_1}] = [\pi_b^E u]_{f_{2j}}, \quad 1 \leq j \leq 4. \tag{5.41}
$$

**Lemma 5.13.** We have the bound

$$su_{i=1}^{4} N_{K_i'}[\mathcal{L}_E^{F_1,K_2}[\pi_b u]]^2 \lesssim p^{10} (N_{K_2} |\eta_b|^2 + \sum_{i=1}^{4} N_{K_i'} |\eta_b|^2). \tag{5.42}
$$

**Proof.** Proceeding as in the proof of (5.12) yields

$$su_{i=1}^{4} N_{K_i'}[\mathcal{L}_E^{F_i,K_2}[\pi_b u]]^2 \lesssim p^4 h^{-1} \sum_{i=1}^{4} \|\pi_b^E u\|_{f_{2i}}^2. \tag{5.42}
$$

Applying the jump estimate (3.9) (with the fact that $[u]_{f_{2i}} = 0$) in conjunction with the bound (5.38) implies the asserted bound.

Next, we introduce the full trace lifting $\mathcal{L}_E^{F_1}[\pi_b u]$ associated with the face $F_1$ as

$$\mathcal{L}_E^{F_1}[\pi_b u] := \begin{cases} \mathcal{L}_E^{F_i}[\pi_b u] + \mathcal{L}_E^{F_i,K_2}[\pi_b u] & \text{on } \{K_1\}_{i=1}^4, \\
0 & \text{otherwise}, \end{cases} \tag{5.43}
$$
contrast to Section 5.3.1, these faces are now regularly matching elemental faces $F$ and $L$. Moreover, since $L_{c}^{E}[\pi_{b}u]$ vanishes on the faces $F_{2}, F_{3}$, the lifting $L_{c}^{E}[\pi_{b}u]$ affects the values of $\pi_{b}u$ on the other faces $F_{1}, F_{2}$ only through the edge lifting $L_{e}^{E}[\pi_{b}u]$, which gives rise to a continuous function in layer $\ell'$ and across patch borders; cp. Remark 5.6.

**Lemma 5.14.** We have $(L_{c}^{E}[\pi_{b}u]|_{K_{i}'})|_{f_{2j}} = \|\pi_{b}u\|_{f_{2j}}$ for $1 \leq j \leq 4$, as well as

$$\sum_{i=1}^{4} N_{K_{i}'}[L_{c}^{E}[\pi_{b}u]]^{2} \leq p^{10} \left( N_{K_{2}'}[\eta_{b}]^{2} + \sum_{i=1}^{4} N_{K_{i}'}[\eta_{b}]^{2} \right).$$

(5.44)

**Proof.** We establish the first assertion for $f_{2j}$ (the proof for the other faces is analogous). By property (5.41) and the definition of $\pi_{b}^{E}u$ in (5.39), we have

$$(L_{c}^{E}[\pi_{b}u]|_{K_{i}'})|_{f_{2j}} = (\mathcal{L}_{c}^{E}[\pi_{b}u]|_{K_{i}'})|_{f_{2j}} + \|\pi_{b}u\|_{f_{2j}} - (\mathcal{L}_{c}^{E}[\pi_{b}u]|_{K_{i}'})|_{f_{2j}} = \|\pi_{b}u\|_{f_{2j}}.$$  

Furthermore, the estimate (5.44) follows with the triangle inequality, by combining the bounds in (5.38) and (5.42).

Upon constructing corresponding face liftings $L_{c}^{E}[\pi_{b}u]$ and $L_{c}^{E}[\pi_{b}u]$ over the faces $F_{2}$ and $F_{3}$, respectively, we finally define the corner lifting $L_{c}^{E}[\pi_{b}u]$ by

$$L_{c}[\pi_{b}u] := \sum_{i=1}^{3} L_{c}^{E}[\pi_{b}u].$$

(5.45)

The lifted interpolant $\Pi u := \pi_{b}u + L_{c}[\pi_{b}u]$ now gives rise to a piecewise polynomial and conforming interpolant across layer $\ell' - 1$ and $\ell'$. The support of the trace lifting does not extend into layer $\ell' + 1$ and preserves homogeneous Dirichlet boundary conditions. With the triangle inequality and bounds as in (5.44), the error $\eta = u - \Pi u$ can be controlled in terms of $\eta_{b} = u - \pi_{b}u$ as follows.

**Lemma 5.15.** We have the bound

$$\sum_{i=1}^{7} (N_{K_{i}'}[\eta]^{2} + N_{K_{i}'}[\eta]^{2}) \leq p^{10} \sum_{i=1}^{7} (N_{K_{i}'}[\eta_{b}]^{2} + N_{K_{i}'}[\eta_{b}]^{2}) .$$

(5.46)

5.3.2. Terminal layer elements in $\tilde{\mathcal{T}}_{\sigma}^{\ell,e}$

We consider the interface from the mesh layer $\mathcal{L}_{c}^{E}[\pi_{b}u] = \{K_{i}\}_{i=1}^{7}$ to the terminal layer $\mathcal{L}_{c}^{E}[\pi_{b}u] = \{K_{i}'\}$ shown in Figure 6. As in (5.31), we denote by $h$ the diameter of the shape-regular elements in the configuration. Analogously to Figure 5, we introduce the elemental faces $F_{1}, F_{2}, F_{3}$ of $K_{1}'$, as well as the elemental edges $E_{1}, \ldots, E_{9}$. In contrast to Section 5.3.1, these faces are now regularly matching elemental faces.
of $K'_1$. Nevertheless, the jumps $[\pi_h u]_{F_i}$ defined as in (5.3) do not vanish in general, since the base projector vanishes on $K'_1$, cp. (4.10). Another consequence of this choice is that properties (5.33), (5.34) are not valid anymore.

To address these difficulties, we introduce polynomial nodal liftings associated with the nodes $N_1, \ldots, N_7$ shown in Figure 6. Here, we emphasize that, under the regularity assumption $u \in M^2_2(\bar{Q})$, the point values $u(N_j)$ are well-defined (since the nodes are separated from the singular corner $e$).

We now focus on node $N_1 = (0,0,0)$. By the nodal exactness (4.4) of $\pi_h u$ in layer $\ell - 1$, we have

$$[\pi_h u]_{F_1}(N_1) = [\pi_h u]_{F_2}(N_1) = [\pi_h u]_{F_3}(N_1) = u(N_1).$$

This prompts us to define the polynomial vertex jump lifting $L^N_{c_1}[\pi_h u]$ by

$$L^N_{c_1}[\pi_h u] := \begin{cases} [\pi_h u]_{F_1}(N_1)L_{N_1}(x) & \text{on } K'_1, \\ 0 & \text{on } K'_{i=1}, \end{cases}$$

with the nodal basis function $L_{N_1}(x) := (1-x_1/a_1)(1-x_2/b_1)(1-x_3/c_1) \in \mathcal{Q}_1(K'_1)$.

**Lemma 5.16.** We have the bounds

$$N_{K'_1}[L^N_{c_1}[\pi_h u]]^2 \lesssim p^h h^{-1} \| [\pi_h u]_{F_1} \|^2_{L^2(F_1)},$$

(5.49)
and

\[ N_{K'_1} |\mathcal{L}_c^{N_1} [\pi_b u]|^2 \lesssim p^8 (N_{K'_1} |\eta_b|^2 + N_{K'_2} |\eta_b|^2). \tag{5.50} \]

**Proof.** As in the proof of Lemma 5.3, we first let the configuration in Figure 3 be of unit size. Clearly,

\[ \| \mathcal{L}_c^{N_1} [\pi_b u] \|_{L^2(K'_1)}^2 \lesssim \| [\pi_b u]_{F_1} (N_1) \|_{L^2(F_1)}^2 \lesssim \| [\pi_b u]_{F_1} \|_{L^2(F_1)}^2. \]

With the inverse inequality in Ref. 16 (Theorem 4.76), we conclude that

\[ \| \mathcal{L}_c^{N_1} [\pi_b u] \|_{L^2(K'_1)}^2 \lesssim \| [\pi_b u]_{F_1} \|_{L^2(F_1)}^2 \lesssim p^8 \| [\pi_b u]_{F_1} \|_{L^2(F_1)}^2. \]

Moreover, applying the inverse inequality (5.14) yields

\[ \| \nabla \mathcal{L}_c^{N_1} [\pi_b u] \|_{L^2(K'_1)} \lesssim p^8 \| [\pi_b u]_{F_1} \|_{L^2(F_1)}, \]

which shows the assertion in the reference case. In the general case, isotropic versions of the scalings in Lemma 3.1 imply the bound (5.49); cp. (5.15) and (5.16). The bound (5.50) follows from (5.49) and the jump bound (3.9) (since \([u]_{F_1} = 0\) for \(F_1 = \text{int}(\partial K_2 \cap \partial K'_1)\)).

Similar constructions for the other nodes in Figure 6 give rise to nodal liftings \(\mathcal{L}_c^{N_j} [\pi_b u]\), for \(1 \leq j \leq 7\), with analogous properties. We then define the full vertex lifting

\[ \mathcal{L}_c^N [\pi_b u] := \sum_{j=1}^7 \mathcal{L}_c^{N_j} [\pi_b u]. \tag{5.51} \]

From bounds analogous to (5.50), we have

\[ N_{K'_1} |\mathcal{L}_c^N [\pi_b u]|^2 \lesssim p^8 (N_{K'_2} |\eta_b|^2 + N_{K'_5} |\eta_b|^2 + N_{K'_7} |\eta_b|^2 + N_{K'_0} |\eta_b|^2). \tag{5.52} \]

**Remark 5.7.** If \(N_j\) corresponds to a boundary node, then the resulting vertex lifting \(\mathcal{L}_c^{N_j} [\pi_b u]\) is identically zero; cp. property (5.47). Moreover, the definitions of corresponding vertex liftings in the terminal layers of adjacent corner or corner-edge patches (matching regularly over patch interfaces due to Assumptions 3.1, 3.2) yield conformity of the nodal liftings \(\mathcal{L}_c^N [\pi_b u]\) across patch borders.

The modified base projector

\[ \pi_b^N u := \begin{cases} \pi_b u + \mathcal{L}_c^N [\pi_b u] & \text{on } K'_1, \\ \pi_b u & \text{otherwise}. \end{cases} \tag{5.53} \]

satisfies the properties analogous to those in Lemma 5.10.

**Lemma 5.17.** We have \([\pi_b^N u]_{F_1} = [\pi_b^N u]_{F_3}\) on \(\overline{E}_1\) and \([\pi_b^N u]_{F_1} (N_j) = 0\) for \(1 \leq j \leq 4\). (Analogous identities hold on the other faces and edges, possibly across patch borders.)
Lemma 5.18. and proves the first assertion. To show (5.56), we note that

\[
\| \pi_b^N u \|_{F_1}(x) = \pi_b|_{K_2} u|_{K_2}(x) - \mathcal{L}_c^N \| \pi_b u \|_{K_1'}(x),
\]
\[
\| \pi_b^N u \|_{F_3}(x) = \pi_b|_{K_3} u|_{K_3}(x) - \mathcal{L}_c^N \| \pi_b u \|_{K_1'}(x).
\]

As in the proof of Lemma 5.10, since \( \{ K_i \} \) match regularly in layer \( \ell - 1 \), we conclude with (4.6) that \( \pi_b|_{K_2} u|_{K_2}(x) = \pi_b|_{K_3} u|_{K_3}(x) \).

Then, by construction and the definition of the jumps we have
\[
\| \pi_b^N u \|_{F_1}(N_j) = \| \pi_b u \|_{F_1}(N_j) - \mathcal{L}_c^N \| \pi_b u \|_{K_1'}(N_j) = \| \pi_b u \|_{F_1}(N_j) - \| \pi_b u \|_{F_1}(N_j) = 0.
\]

This yields the second assertion on the face \( F_1 \).

With Lemma 5.17, we now continue our analysis along the lines of Section 5.3.1, by employing the projector \( \pi_b^N u \) instead of \( \pi_b u \). To do so, we first introduce edge liftings associated with the edge jumps \( \| \pi_b^N u \|_{F_i} \) and defined similarly to those in (5.17), (5.35), (5.37). Second, we adapt the polynomial face lifting \( \mathcal{L}_c^{K_i'}[\cdot] \) in (5.43) (and analogously \( \mathcal{L}_c^{K_i'}[\cdot], \mathcal{L}_c^{K_i'}[\cdot] \)) to the slightly different setting in Figure 6, giving rise to a polynomial corner lifting \( \mathcal{L}_c^{K_i'}[\pi_b^N u] \) as in (5.45), (5.46), but defined only over \( K_i' \). By construction, this lifting satisfies

\[
\left(\mathcal{L}_c^{K_i'}[\pi_b^N u]|_{K_i'}\right)|_{F_j} = \| \pi_b^N u \|_{F_j}, \quad 1 \leq j \leq 3.
\]

In the configuration of Figure 6, we introduce the terminal layer lifting

\[
\mathcal{L}_c^T[\pi_b u] := \mathcal{L}_c^N[\pi_b u] + \mathcal{L}_c^{K_i'}[\pi_b^N u].
\]

Lemma 5.18. We have \( \mathcal{L}_c^T[\pi_b u]|_{K_i'}\rangle_{F_j} = \| \pi_b u \|_{F_j} \) for \( 1 \leq j \leq 3 \), as well as

\[
N_{K_i'}[\| \mathcal{L}_c^T[\pi_b u] \|_{K_i'}^2] \lesssim p^{18} \left( N_{K_i'}[\| \pi_b \|^2] + N_{K_2}[\| \psi \|^2] + N_{K_3}[\| \psi \|^2] + N_{K_i'}[\| \psi \|^2] \right).
\]

Proof. We apply identity (5.54) and the definition of the jumps. This results in

\[
\mathcal{L}_c^T[\| \pi_b u \|_{K_i'}]\rangle_{F_j} = \mathcal{L}_c^T[\| \pi_b u \|_{K_i'}]\rangle_{F_j} + \| \pi_b u \|_{F_j} - \left( \mathcal{L}_c^T[\| \pi_b u \|_{K_i'}]\rangle_{F_j} = \| \pi_b u \|_{F_j},
\]

and proves the first assertion. To show (5.56), we note that

\[
N_{K_i'}[\| \mathcal{L}_c^T[\pi_b u] \|_{K_i'}^2] \lesssim N_{K_i'}[\| \mathcal{L}_c^T[\pi_b u] \|_{K_i'}^2] + N_{K_i'}[\| \mathcal{L}_c^{K_i'}[\pi_b^N u] \|_{K_i'}^2].
\]

According to (5.52), we have

\[
N_{K_i'}[\| \mathcal{L}_c^T[\pi_b u] \|_{K_i'}^2] \lesssim p^8 \left( N_{K_i'}[\| \psi \|^2] + N_{K_2}[\| \psi \|^2] + N_{K_3}[\| \psi \|^2] + N_{K_i'}[\| \psi \|^2] \right).
\]

Moreover, by proceeding as in the proof of (5.44) we find that

\[
N_{K_i'}[\| \mathcal{L}_c^{K_i'}[\pi_b^N u] \|_{K_i'}^2] \lesssim p^{10} \left( N_{K_2}[\| \psi \|^2] + N_{K_3}[\| \psi \|^2] + N_{K_i'}[\| \psi \|^2] + N_{K_1}[\| u - \pi_b^N u \|]^2 \right).
\]

Finally, with the triangle inequality and as above with (5.52),

\[
p^{10}N_{K_i'}[u - \pi_b^N u]^2 \lesssim p^{18} \left( N_{K_i'}[\| \psi \|^2] + N_{K_2}[\| \psi \|^2] + N_{K_3}[\| \psi \|^2] + N_{K_i'}[\| \psi \|^2] \right)
\]

This completes the proof.
With Lemma 5.18 the lifted interpolant \( \Pi_{\ell} := \pi_{\ell} u + \mathcal{P}_0^\ell |\pi_{\ell} u| \) is conforming across layers \( \ell - 1 \) and \( \ell \), and preserves the essential boundary conditions. Analogously to Section 5.2.2 and with the estimate (5.56), the terminal layer version of Lemma 5.15 follows.

**Lemma 5.19.** We have the bound
\[
N_{K_i}^\ell [\eta]^2 + \sum_{i=1}^{7} N_{K_i}^{\ell-1} [\eta]^2 \lesssim p^{18} \left( N_{K_i}^\ell [\eta_b]^2 + \sum_{i=1}^{7} N_{K_i}^{\ell-1} [\eta_b]^2 \right) .
\] (5.57)

This completes the construction of the jump liftings for the corner patch \( \hat{\mathcal{M}}_{\sigma}^{\ell,ce} \). Lemma 5.15 (for submesh \( \hat{\mathcal{D}}_{\sigma}^{\ell,ce} \)) and Lemma 5.19 (for submesh \( \hat{\mathcal{T}}_{\sigma}^{\ell,ce} \)) give the bound (3.26) over the reference corner patch \( \hat{\mathcal{M}}_{\sigma}^{\ell,ce} \).

**Proposition 5.3.** With (5.1) we have \( \chi_{\hat{\mathcal{M}}_{\sigma}^{\ell,ce}}[\eta] \lesssim p^{18} \chi_{\hat{\mathcal{M}}_{\sigma}^{\ell,ce}}[\eta_b] \).

### 5.4. Corner-edge patch \( \hat{\mathcal{M}}_{\sigma}^{\ell,ce} \)

We finally construct polynomial jump liftings in the reference corner-edge patch \( \hat{\mathcal{M}}_{\sigma}^{\ell,ce} \). We consider in detail the case of a corner-edge patch with geometric refinements along a single edge \( e \) as shown in Figure 1 (right), and comment on the case with refinements along two or three edges in Section 5.4.2 ahead.

#### 5.4.1. Refinement along one edge

Let \( \hat{\mathcal{M}}_{\sigma}^{\ell,ce} \) be a corner-edge reference patch mesh on \( \bar{Q} \) with geometric refinement along the single edge \( e \). We partition the patch mesh into
\[
\hat{\mathcal{M}}_{\sigma}^{\ell,ce} := \hat{\mathcal{T}}_{\sigma}^{\ell,ce} \cup \hat{\mathcal{D}}_{\sigma}^{\ell,ce} \cup \hat{\mathcal{M}}_{\sigma}^{\ell,ce} , \quad \ell \geq 1 ,
\] (5.58)
where \( \hat{\mathcal{T}}_{\sigma}^{\ell,ce} \) consists of a single and isotropic terminal layer element at the corner \( c \), and the mesh \( \hat{\mathcal{D}}_{\sigma}^{\ell,ce} \) is a corner-patch type mesh consisting of shape-regular elements which are refined into the corner \( c \). The mesh \( \hat{\mathcal{M}}_{\sigma}^{\ell,ce} \) consists of a sequence of \( \ell \) geometrically scaled edge-patch meshes, properly translated along the edge \( e \):
\[
\hat{\mathcal{M}}_{\sigma}^{\ell,ce} = \bigcup_{\ell' = 1}^{\ell} \hat{\Psi}^{\ell,ce}(\hat{\mathcal{M}}_{\sigma}^{\ell',e}) ,
\] (5.59)
where we denote by \( \hat{\Psi}^{\ell,ce} \) the operation of translation with respect to the edge-parallel variable \( x^\parallel \) combined with a dilation by a factor only depending on \( \sigma, \ell, \ell' \), and where the mesh \( \hat{\mathcal{M}}_{\sigma}^{\ell,ce} \) is a reference edge patch on \( \bar{Q} \) with \( \ell' + 1 \) mesh layers; cp. (3.5). In Figure 7, a schematic illustration of the patch decomposition (5.58), (5.59) is provided.

The two outermost edge-patch meshes \( \hat{\Psi}^{\ell-1,ce}(\hat{\mathcal{M}}_{\sigma}^{\ell-1,ce}) \) and \( \hat{\Psi}^{\ell,ce}(\hat{\mathcal{M}}_{\sigma}^{\ell,ce}) \) are indicated in Figure 8 (left). There, the hexahedra \( K_3, K_4, K_6 \) in \( \hat{\Psi}^{\ell-1,ce}(\hat{\mathcal{M}}_{\sigma}^{\ell-1,ce}), \)
Fig. 7. Patch decomposition (5.58), (5.59) for \( \sigma = 0.5 \) and \( \ell = 5 \). The scaled edge-patch meshes for \( \ell' = 1, \ldots, \ell \) are displayed in boldface. The subdiagonal elements in \( D_{\ell'} \) are shaded. The single terminal layer element in \( \tilde{\Omega}_{0,\ell}^x \) is indicated. The remaining elements belong to the corner-type mesh \( D_{\ell}^{x,c,1} \).

Fig. 8. Left: The edge-patch meshes \( \tilde{\Omega}_{0,\ell}^{x,c}(\tilde{\Omega}_{0,\ell}^{x,c}) \) and \( \tilde{\Omega}_{\ell}^{x,c}(\tilde{\Omega}_{\ell}^{x,c}) \) for \( \sigma = 0.5 \) and \( \ell = 5 \). Right: Interface between \( \tilde{\Omega}_{0,\ell}^{x,c} = \{ K'_{1} \} \) and the first edge-patch block \( \tilde{\Omega}_{0,\ell}^{x,c}(\tilde{\Omega}_{0,\ell}^{x,c}) \) for \( \sigma = 0.5 \) and \( \ell = 2 \).

and \( K_{3}, K_{4}, K_{6} \) in \( \tilde{\Omega}_{0,\ell}^{x,c}(\tilde{\Omega}_{0,\ell}^{x,c}) \) will be referred to as \textit{diagonal elements}. Moreover, the immediate neighbors of \( K_{3}', K_{4}', K_{6}' \) across the patch interface are denoted by

\[
D_{\ell} := \{ K_{3}'', K_{4}'', K_{6}''' \},
\]

and are called the \textit{subdiagonal elements}. (Note that \( K_{3}, K_{3}', K_{3}'' \) are not explicitly labelled in Figure 8.) The diagonal and subdiagonal elements are shape-regular (uniformly in \( \ell \)). We may thus assume that

\[
h_{K_{i}} \simeq h_{K_{i}'} \simeq h_{K_{i}''} \simeq h, \quad i \in \{3, 4, 6\},
\]

(5.61)
for a mesh size parameter $h$. Corresponding conventions will be employed in the other scaled edge-patch meshes for $2 \leq \ell' \leq \ell$. In particular, we denote by $D_{\ell'}$ the corresponding set of subdiagonal elements. In Figure 8 (right), we further illustrate the interface between $\widetilde{T}_{\sigma}^{\ell,e} = \{K'_1\}$ and $\widetilde{\Psi}^{1,ce}(\widetilde{M}_{\sigma}^{1,e})$. The diagonal elements are $K_3, K_4, K_6$, while the set

$$D_1 := \{K''_1\}$$

(5.62)

consists of a single subdiagonal element in $\widetilde{\Psi}^{1,ce}(\widetilde{M}_{\sigma}^{1,e})$.

In the sequel, we construct polynomial jump liftings over irregular faces of $\widetilde{M}_{\sigma}^{\ell,ce}$ in (5.58), (5.59) starting from the base projector $\pi_b u$ defined in (4.10). First, on all irregular faces which are situated parallel to $e$ and which belong to an edge-patch mesh $\widetilde{\Psi}^{\ell,ce}(\widetilde{M}_{\sigma}^{\ell,e})$, $1 \leq \ell' \leq \ell$, we apply the (scaled) jump liftings $\mathcal{L}_e[\pi_b u]$ and $\mathcal{L}_e^T[\pi_b u]$ constructed in (5.8) and (5.17), respectively. For $\ell' = 1$, only a scaled version of $\mathcal{L}_e^T[\pi_b u]$ is employed. As shown in Lemma 5.2 (for $\mathcal{L}_e[\pi_b u]$) and Lemma 5.8 (for $\mathcal{L}_e^T[\pi_b u]$), these liftings remove the polynomial face jumps across irregular faces situated as in Figures 2 and 3. We denote by $\pi_e^* u$ the intermediate base projector thus constructed from $\pi_b u$ over $\widetilde{M}_{\sigma}^{1,ce}$; this yields a continuous projector within each edge-patch block.

**Lemma 5.20.** We have $\mathcal{Y}_{\widetilde{M}_{\sigma}^{\ell,ce}}[u - \pi^*_e u] \lesssim p^{10} \mathcal{Y}_{\widetilde{M}_{\sigma}^{\ell,ce}}[u - \pi_b u]$.

**Proof.** This follows from the bound in Proposition 5.2, which holds true for edge-parallel hexahedra of general aspect ratio with $0 < h^{\perp} \lesssim h^{\parallel} \lesssim 1$.

We further analyze the continuity properties of $\pi^*_e u$ over $\widetilde{M}_{\sigma}^{\ell,ce}$ in edge-parallel direction (i.e., across interfaces perpendicular to $e$). As in Remarks 5.2 and 5.5, the tensor-product structure of the projector $\pi_b u$ and the liftings $\mathcal{L}_e[\pi_b u]$, $\mathcal{L}_e^T[\pi_b u]$ ensures the continuity of the intermediate projection $\pi^*_e u$ across the scaled edge-patch mesh $\widetilde{\Psi}^{\ell,ce}(\widetilde{M}_{\sigma}^{\ell,e})$ into an adjacent edge or corner-edge patch where the same jump liftings are applied. Then, to investigate the conformity between adjacent edge-patch blocks across interfaces perpendicular to $e$, we may again focus on the two adjacent edge-patch blocks $\widetilde{\Psi}^{\ell-1,ce}(\widetilde{M}_{\sigma}^{\ell-1,e})$, $\widetilde{\Psi}^{\ell,ce}(\widetilde{M}_{\sigma}^{\ell,e})$ shown in Figure 8 (left). Then, with the same arguments as above, $\pi^*_e u$ is continuous across such interfaces, with the exception of the perpendicular faces associated with the subdiagonal elements $K''_3$, $K''_4$, $K''_6$ of $\widetilde{\Psi}^{\ell,ce}(\widetilde{M}_{\sigma}^{\ell,e})$. Indeed, by definition of the liftings $\mathcal{L}_e[\pi_b u]$ in (5.8),

$$\pi^*_e u = \pi_b u \quad \text{on } \{K_3, K'_3, K_4, K'_4, K_6, K''_6\},$$

(5.63)

since the diagonal elements constitute the outermost layers in the scaled edge-patch blocks $\widetilde{\Psi}^{\ell-1,ce}(\widetilde{M}_{\sigma}^{\ell-1,e})$ and $\widetilde{\Psi}^{\ell,ce}(\widetilde{M}_{\sigma}^{\ell,e})$. However, on the subdiagonal elements $K''_3$, $K''_4$, $K''_6$ in $\widetilde{\Psi}^{\ell,ce}(\widetilde{M}_{\sigma}^{\ell,e})$ depicted in Figure 8 (left), the base projector $\pi_b u$ is altered by the addition of the (scaled) lifting $\mathcal{L}_e[\pi_b u]$ which does not generally vanish on the interface $x_3 = 0$; cp. Remark 5.2. As a consequence, the projector $\pi^*_e u$ is generally not continuous over the faces $f_{K''_3}, K''_3$, $f_{K''_4}, K''_4$, $f_{K''_6}, K''_6$. By the self-similar structure of the mesh $\widetilde{M}_{\sigma}^{\ell,ce}$, the same difficulty arises in the subdiagonal
elements of each scaled edge patch $\tilde{\Psi}^{\ell,ce}(\tilde{M}^{\ell,e}_\sigma)$, for $2 \leq \ell' \leq \ell$, cp. Figure 7. In addition, in the case $\ell' = 1$, see Figure 8 (right), it follows similarly that

$$\pi^{\ell}_b u = \pi^{\ell}_b u \quad \text{on } \{K_3, K_4, K_6, K_1'\},$$  \hspace{1cm} (5.64)

and that $\pi^{\ell}_b u$ is in general non-conforming over the face $f_{K_1', K_1''}$.

Second, we correct the non-conformity across the critical faces as discussed above. We consider first the case $2 \leq \ell' \leq \ell$. In Figure 9, we illustrate the three subdiagonal elements $K''_3, K''_4, K''_6$ on the patch interface between $\tilde{\Psi}^{\ell',ce}(\tilde{M}^{\ell-1,e}_\sigma)$ and $\tilde{\Psi}^{\ell,ce}(\tilde{M}^{\ell,e}_\sigma)$, employing the notation in Figure 8 (left). Without loss of generality, we may assume that the interface lies on $x_3 = 0$.

![Figure 9. Subdiagonal elements $K''_3, K''_4, K''_6$ on the interface $x_3 = 0$ between two adjacent edge-patch blocks for $\sigma = 0.5$ and $2 \leq \ell' \leq \ell$. The nodes $N_3, N_4, N_6$ and the edges $E_1, E_2$ are indicated.](image-url)

**Lemma 5.21.** Let $2 \leq \ell' \leq \ell$. In the setting of Figure 9, we have

$$\pi^{\ell}_b|_{K_i'}u|_{K_i'}(N_i) = \pi^{\ell}_b|_{K''_i}u|_{K''_i}(N_i), \quad i \in \{3, 4, 6\},$$  \hspace{1cm} (5.65)

for the nodes $N_3 = (0, 0, 0), N_4 = (a, 0, 0), N_6 = (0, b, 0)$, as well as

$$\pi^{\ell}_b|_{K_i'}u|_{K_i'} = \pi^{\ell}_b|_{K''_i}u|_{K''_i}$$  \hspace{1cm} (5.66)

on each closed line segment shown in boldface for the corresponding element index $i \in \{3, 4, 6\}$. In addition, on the edge $E_i = E_1 \cap \partial K_i'$ with $i \in \{3, 4\}$, there holds

$$\left(\pi^{\ell}_b|_{K_i'}u|_{K_i'}\right)|_{E_i} = \left(\pi^{\ell}_b|_{K''_i}u|_{K''_i}\right)|_{E_i} = \left(\pi_b|_{K_i'}u|_{K_i'}\right)|_{E_i}.$$  \hspace{1cm} (5.67)

Similarly, on the edge $E_i = E_2 \cap \partial K_i'$ with $i \in \{3, 6\}$, there holds

$$\left(\pi^{\ell}_b|_{K_i'}u|_{K_i'}\right)|_{E_i} = \left(\pi_b|_{K_i'}u|_{K_i'}\right)|_{E_i} = \left(\pi^{\ell}_b|_{K''_i}u|_{K''_i}\right)|_{E_i}.$$  \hspace{1cm} (5.68)

**Proof.** We verify (5.65) for $i = 3$. With the properties (5.63), (4.6), the definition of $\pi^{\ell}_b u$ and the identities (5.6), we conclude that

$$\pi^{\ell}_b|_{K''_3}u|_{K''_3}(N_3) = \pi_b|_{K''_3}u|_{K''_3}(N_3) = \pi^{\ell}_b|_{K''_3}u|_{K''_3}(N_3) = \pi^{\ell}_b|_{K''_3}u|_{K''_3}(N_3).$$
Moreover, consider a line segment and the corresponding index $i \in \{3, 4, 6\}$. Hence, by Lemma 5.2, $\pi_b^i|_{K''} u|_{K''}$ is identically equal to $\pi_b|_{K''} u|_{K''}$ on the segment, which in turn is equal to $\pi_b|_{K'} u|_{K'} = \pi_b^i|_{K'} u|_{K'}$; cp. property (4.6). This yields (5.66). The properties (5.67) and (5.68) follow from (5.63) and the fact that $\pi_b u$ is continuous over $f_{K_2, K_4}$ and $f_{K_6, K_6}$.

We next adjust the projector $\pi_b^i u$ on the faces of the subdiagonal elements of $D'$ in (5.60). To that end, for an axiparallel hexahedral element $K$, we will express $\pi_b^i|_{K} u|_{K}$ as a linear combination of the Lagrangian basis functions associated with the (mapped) $(p+1)$-point tensor-product Gauss-Lobatto interpolation nodes on $K$; cp. Refs. 3, 18. Fix an element $K'' \in D'$ with $i \in \{3, 4, 6\}$. We denote by $f_i$ the face $f_{K''}$. On $K'', \pi_b^i u$ to $\pi_b^{D_i} u$, by explicitly altering the $(p+1)^2$ Gauss-Lobatto face degrees of freedom of $(\pi_b^i u)|_{K''}$ on $f_i \subset \partial K''$ so that we have

$$(\pi_b^{D_i}|_{K''} u|_{K''})|_{f_i} = (\pi_b^i|_{K''} u|_{K''})|_{f_i}, \quad i \in \{3, 4, 6\}. \quad (5.69)$$

In view of Lemma 5.21, the projector $\pi_b^{D_i} u$ yields a conforming approximation, which is now continuous across the critical faces $f_i = \overline{f_{K''}}$ for $i \in \{3, 4, 6\}$ and has the same continuity properties as $\pi_b^i u$ elsewhere. It also preserves homogeneous Dirichlet boundary conditions.

**Lemma 5.22.** Let $2 \leq \ell' \leq \ell$. Then:

$$\pi_b^{D_i} u = \pi_b^i u = \pi_b u \quad \text{on } \{K_3, K_3', K_4, K_4', K_6, K_6'\}. \quad (5.70)$$

In addition, we have the bounds

$$N_{K''} [\pi_b^{D_i} u - \pi_b^i u]^2 \lesssim p^2 h^{-1} \|\pi_b^{D_i} u - \pi_b^i u\|^2_{L^2(f_i)}, \quad i \in \{3, 4, 6\}, \quad (5.71)$$

with the mesh size parameter $h$ in (5.61), and

$$\sum_{i \in \{3, 4, 6\}} N_{K''} [\pi_b^{D_i} u - \pi_b^i u]^2 \lesssim p^2 \sum_{i \in \{3, 4, 6\}} (N_{K''} |u - \pi_b^i u|^2 + N_{K''} |u - \pi_b^i u|^2). \quad (5.72)$$

**Proof.** Property (5.70) follows from (5.63) and (5.69). To verify (5.71), we note that, upon an isotropic scaling argument as in Lemma 3.1 and (5.15), (5.16), it is sufficient to consider the case where $K''$ and $K_i'$ are of unit size. By (5.69), the difference $(\pi_b^{D_i} u - \pi_b^i u)|_{K''}$ then vanishes in the interior Gauss-Lobatto tensor-product points on $K''$. The inequality in Ref. 3 (Lemma 3.1, Equation (16)) thus yields

$$\|\pi_b^{D_i} u - \pi_b^i u\|^2_{L^2(K''')} \lesssim p^2 \|\pi_b^{D_i} u - \pi_b^i u\|^2_{L^2(f_i)}.$$

Again, the inverse inequality (5.14) shows that

$$\|\nabla (\pi_b^{D_i} u - \pi_b^i u)\|^2_{L^2(K''')} \lesssim p^2 \|\pi_b^{D_i} u - \pi_b^i u\|^2_{L^2(f_i)},$$

which implies (5.71).
To establish (5.72), we start from the bound (5.71) and employ the triangle inequality (along with the fact that \( u \) is continuous over \( f_1 \)), property (5.69), and the trace inequality in Ref. 14 (Lemma 4.2 with \( t = 2 \)); cp. the bound (3.9). This readily results in

\[ N_{K'_i}[\pi_b^{D,*}u - \pi_b^*u]^2 \lesssim p^2 h^{-1}(\|u|_{K'_i} - (\pi_b^{D,*}u)|_{K'_i}\|_{L^2(f_1)}^2 + \|(u - \pi_b^*u)|_{K'_i}\|_{L^2(f_1)}^2) \]

\[ \lesssim p^2 h^{-1}(\|u - \pi_b^*u\|_{K'_i}\|_{L^2(f_1)}^2 + \|(u - \pi_b^*u)|_{K'_i}\|_{L^2(f_1)}^2) \]

\[ \lesssim p^2 (N_{K'_i}[u - \pi_b^*u]^2 + N_{K'_i}[u - \pi_b^*u]^2). \]

The assertion follows.

\(\square\)

The case \( \ell' = 1 \) is treated analogously; cp. Figure 8 (right). In accordance to (5.69), we define \( \pi_b^{D,*}u \) on \( K'_i \in \mathcal{D}_1 \) so that

\[ (\pi_b^{D,*}|_{K'_i}|u|_{K'_i}) \big|_{T'} = (\pi_b^*|_{K'_i}|u|_{K'_i}) \big|_{T'}, \quad (5.73) \]

over the face \( f_1 = f_{K'_i,K'_i} \). The projector \( \pi_b^{D,*}u \) is conforming over the face \( f_{K'_i,K'_i} \) in Figure 8 (right) and preserves potential homogeneous Dirichlet boundary conditions on patch faces. The analog of (5.72) reads as follows.

**Lemma 5.23.** Let \( \ell' = 1 \). Then:

\[ \pi_b^{D,*}u = \pi_b^*u = \pi_b^*u \quad \text{on} \quad \{K_3, K_4, K_6, K'_1\}. \]

In addition, we have the bound

\[ N_{K'_i}[\pi_b^{D,*}u - \pi_b^*u]^2 \lesssim p^2 (N_{K'_i}[u - \pi_b^*u]^2 + N_{K'_i}[u - \pi_b^*u]^2). \]

In conclusion, the projector \( \pi_b^{D,*}u \) defined in (5.69), (5.73) gives a piecewise polynomial and conforming approximation over the entire mesh \( \mathcal{M}^{\ell,ce}_\sigma \) in (5.59).

In addition, it is conforming over the interfaces between \( \mathcal{M}^{\ell,ce}_\sigma \) and \( \mathcal{M}^{\ell,ce}_\sigma \) in edge-parallel direction (i.e., over the face \( f_{K'_i,K'_i} \) in Figure 8 (right)). The projector \( \pi_b^{D,*}u \) satisfies homogeneous Dirichlet boundary conditions on corresponding patch boundaries.

**Lemma 5.24.** We have \( \mathcal{Y}_{\mathcal{M}^{\ell,ce}_\sigma}[u - \pi_b^{D,*}u] \lesssim p^{12} \mathcal{Y}_{\mathcal{M}^{\ell,ce}_\sigma}[u - \pi_b^*u] \).

**Proof.** By employing the triangle inequality, this follows from Lemmata 5.22, 5.23 in combination with Lemma 5.20.

Third, we address the continuity across faces between \( \tilde{\mathcal{Y}}_{\mathcal{M}^{\ell',ce}_\sigma}(\tilde{\mathcal{M}}^{\ell',ce}_\sigma) \) and \( \tilde{\mathcal{D}}_{\sigma',e,1}^{\ell',e,1} \) for \( 1 \leq \ell' \leq \ell \). We consider again the two configurations in Figure 8. In the case \( 2 \leq \ell' \leq \ell \), the projector \( \pi_b^{D,*}u \) is already continuous over the regular faces \( f_{K_1,K_3}, f_{K_2,K_4}, \) and \( f_{K_5,K_6} \), due to properties (5.70) and (4.6). Similarly, \( \pi_b^{D,*}u \) is conforming over the faces \( f_{K'_1,K'_1}, f_{K'_2,K'_4}, f_{K'_3,K'_6} \). For \( \ell' = 1 \), continuity of \( \pi_b^{D,*}u \) is ensured across \( f_{K_1,K_3}, f_{K_2,K_4} \) and \( f_{K_5,K_6} \); cp. (5.74). However, there are in general nonzero polynomial face jumps across the irregular faces \( f_{2i} = f_{K_2,K'_i}, i \in \{1, 2, 3, 4\} \), and
$f_{b_i} = f_{K_i,K'_i}$ for $i \in \{1,4,5,6\}$, as well as over analogous faces in the omitted mesh structure (recall from (5.70) that $\pi_{b_i}^{D,*} = \pi_{b_i}$ on $K'_3, K'_4, K'_5$). The geometric situation is similar to that in the shape-regular corner patch $\mathcal{M}_{\sigma}^{\ell,ce}$ shown in Figure 4. Over these irregular faces, we thus construct polynomial face jump liftings similarly to $\mathcal{L}_c[\pi_b u]$ in (5.43), (5.45) (with the aid of associated edge liftings as in (5.35), (5.37)). The face liftings are supported in $K'_1 \cup K'_2 \cup K'_3 \cup K'_4$ and $\{K'_1, K'_2, K'_3, K'_4\}$ and are stable polynomially in $p$, as was shown in Section 5.3. We also note that conformity across the corner-edge patch into a neighboring corner or corner-edge patch is guaranteed by Assumptions 3.1, 3.2. We denote by $\pi_{b_i}^{D,*}$ the lifted projector thus obtained from the base projector $\pi_{b_i}$ on $\tilde{\mathcal{D}}_{\sigma,\ell,ce}$. Lemma 5.15 then implies the ensuing bound.

**Lemma 5.25.** We have $\Upsilon_{\tilde{\mathcal{D}}_{\sigma,\ell,ce}} [u - \pi_{b_i}^{D,*} u] \lesssim p^{10} \Upsilon_{\tilde{\mathcal{D}}_{\sigma,\ell,ce}} [u - \pi_{b_i} u]$.

Fourth, we enforce continuity into $\tilde{\mathcal{D}}_{\sigma,\ell,ce}$ in (5.58). The projector $\pi_{b_i}^{D,*}$ over $\tilde{\mathcal{M}}_{\sigma,\ell,ce}$ is continuous across $f_{K'_i,K'_j}$ in Figure 8 (right). Hence, it remains to enforce the conformity of $\pi_{b_i}^{D,*} u$ over $\tilde{\mathcal{D}}_{\sigma,\ell,ce}$ from $K_2, K_3$ into the corner element $K'_4$. Since $\pi_{b_i}^{D,*} = \pi_{b_i} u$ on the faces $f_{K_{i,1},K_{i,2}}$, this can be achieved with the introduction of face jump liftings for $\pi_{b_i}^{D,*}$ over $f_{K_{i,1},K_{i,2}}$ and $f_{K_{i,1},K_{i,3}}$, similarly to $\mathcal{L}_e[\pi_{b_i} u]$ in (5.55) and in conjunction with vertex and edge jump liftings as in Section 5.3.2 (which yield conforming approximations across neighboring patches). A bound analogous to (5.57) holds for the lifted projector still denoted by $\pi_{b_i}^{D,*}$, with an algebraic loss of $p^{18}$.

**Lemma 5.26.** We have $\Upsilon_{\tilde{\mathcal{D}}_{\sigma,\ell,ce}} [u - \pi_{b_i}^{D,*} u] \lesssim p^{18} \Upsilon_{\tilde{\mathcal{D}}_{\sigma,\ell,ce}} [u - \pi_{b_i} u]$.

We define the lifted patch projector $\Pi u$ over the corner-edge patch $\tilde{\mathcal{M}}_{\sigma,\ell,ce}$ as $\Pi u := \pi_{b_i}^{D,*} u$ on $\tilde{\mathcal{M}}_{\sigma,\ell,ce}$ and $\Pi u := \pi_{b_i}^{D,*} u$ on $\tilde{\mathcal{D}}_{\sigma,\ell,ce}$. It is conforming over the entire corner-edge patch and preserves essential boundary conditions. Moreover, Assumptions 3.1, 3.2 ensure the conformity over adjacent patches. With the bounds in Lemmas 5.24, 5.25 and 5.26, we conclude that bound (3.26) holds over the corner-edge patch $\tilde{\mathcal{M}}_{\sigma,\ell,ce}$.

**Proposition 5.4.** With (5.1) we have $\Upsilon_{\tilde{\mathcal{M}}_{\sigma,\ell,ce}} [h] \lesssim p^{18} \Upsilon_{\tilde{\mathcal{M}}_{\sigma,\ell,ce}} [\eta_h]$.

This completes the construction of the reference corner-edge patch projector in the case of refinement along one edge $e$.

5.4.2. Simultaneous refinements

Finally, we comment on extensions to the case where $\tilde{\mathcal{M}}_{\sigma,\ell,ce}$ is simultaneously refined along two or three edges.

We first consider refinements towards two edges $e_1, e_2$ as shown in Figure 10. In this situation and analogously to (5.58), we write

$$\tilde{\mathcal{M}}_{\sigma,\ell,ce} := \tilde{\mathcal{D}}_{\sigma,\ell,ce} \cup \tilde{\mathcal{D}}_{\sigma,\ell,ce_1} \cup \left( \tilde{\mathcal{M}}_{\sigma,\ell,ce_1,1} \cup \tilde{\mathcal{M}}_{\sigma,\ell,ce_1,2} \right),$$

(5.76)
where \( \tilde{\mathcal{T}}_\sigma^f,e \) consists again of the single terminal layer element abutting at the corner \( \mathbf{e} \), and \( \tilde{\mathcal{D}}_{\sigma,x_3}^f \) is a corner-type mesh in \( x_3 \)-direction. The submeshes \( \bar{\mathcal{M}}_{\sigma}^f,e_{1:1} \) and \( \tilde{\mathcal{M}}_{\sigma}^f,e_{2:2} \) are two non-disjoint sequences of \( f \) scaled edge-patch meshes as in (5.59); they overlap over the mutual diagonal elements \( K_3, K_6 \) and \( K'_3, K'_6 \) as illustrated in Figure 10 (left) (the overlap in the omitted mesh structure is analogous). In the sub-

![Fig. 10. Refinement towards two edges. Left: Outermost edge-patch blocks for \( \sigma = 0.5 \) and \( \ell = 5 \). Right: Interface between \( \mathcal{T}_\sigma^f = \{ K'_1 \} \) and the first edge-patch blocks for \( \sigma = 0.5 \) and \( \ell = 2 \).](image)

meshes \( \bar{\mathcal{M}}_{\sigma}^f,e_{1:1} \) and \( \tilde{\mathcal{M}}_{\sigma}^f,e_{2:2} \), we apply the liftings \( \mathcal{L}_e[\pi_{b,u}], \mathcal{L}_e^T[\pi_{b,u}] \) in (5.8), (5.28), respectively. As in (5.63), these liftings leave the base projector \( \pi_{b,u} \) unchanged on the diagonal elements, which are given by \( K_1, K_3, K_4, K_6 \) and \( K'_1, K'_3, K'_4, K'_6 \) for blocks away from \( \mathbf{e} \) (case \( 2 \leq \ell' \leq \ell \)) and by \( K_1, K_3, K_4, K_6 \) and \( K'_1 \) for the last configuration abutting at \( \mathbf{e} \) (case \( \ell' = 1 \)). Similarly to the discussion in Section 5.4.1, the resulting lifted projector \( \pi_{b,u}^T \) is generally non-conforming over the regular faces \( f_{K'_1,K''_1}, f_{K'_2,K''_2}, f_{K'_3,K''_3}, f_{K'_4,K''_4}, f_{K'_5,K''_5}, f_{K'_6,K''_6} \), and \( f_{K'_1,K''_1} \) in the case of edge-patch block \( \ell' = 2 \), as well as over \( f_{K'_1,K''_1} \) and \( f_{K'_2,K''_2} \) in the case \( \ell' = 1 \). (Note that the elements \( K_3, K'_3, K''_3, K'''_3 \) are not explicitly labelled in Figure 10.)

The definitions (5.69), (5.73) of the projector \( \pi_{b,u}^T \) can be readily extended to the configuration in Figure 10, by replacing the set \( \mathcal{D}_V \) in (5.63), (5.64) by

\[
\mathcal{D}_V = \begin{cases} 
\{ K''_1, K''_2, K''_3, K''_4, K''_5, K''_6 \}, & 2 \leq \ell' \leq \ell, \\
\{ K''_1, K''_2 \}, & \ell' = 1,
\end{cases}
\]

where we employ notation as in Figure 10. The projector \( \pi_{b,u}^T \) coincides with \( \pi_{b,u} \) on the diagonal elements indicated in Figure 10; cp. properties (5.70), (5.74). Moreover, corresponding variants of Lemma 5.22, Lemma 5.23 and Lemma 5.24 hold true. Continuity across faces between \( \tilde{\mathcal{D}}_{\sigma,x_3}^f \) and \( \bar{\mathcal{M}}_{\sigma}^f,e_{1:1} \), \( \tilde{\mathcal{M}}_{\sigma}^f,e_{2:2} \) follows as in
Section 5.4.1, by invoking property (4.6) across regular faces. Polynomial jumps over irregular faces within $\mathcal{O}_{e}^{\cdot,ce}$ are lifted as in the corner patch case, and the corner element $K'_1$ in $\tilde{\mathcal{X}}_{e}^{\ell}$ is treated as before, thereby providing estimates as in Lemmas 5.25 and 5.26. Consequently, Proposition 5.4 holds true also for corner-edge patches which are refined along two edges.

Next, we consider the case of refinements towards the three edges $e_1, e_2, e_3$ as illustrated in Figure 11. In this case, we decompose the corner-edge patch into

$$\tilde{\mathcal{M}}^{\ell,ce}: = \mathcal{F}_{e}^{\sigma} \cup (\tilde{\mathcal{M}}_{\sigma}^{\ell,ce_1} : \| \cup \tilde{\mathcal{M}}_{\sigma}^{\ell,ce_2} : \| \cup \tilde{\mathcal{M}}_{\sigma}^{\ell,ce_3} : \|).$$

(5.78)

The set $\tilde{\mathcal{X}}_{e}^{\ell}$ contains the single corner element, and $\tilde{\mathcal{M}}_{\sigma}^{\ell,ce_1} : \|$ is sequence of $\ell$ scaled edge-patch blocks as in (5.59) along edge $e_i$. These sequences overlap on the mutual diagonal elements as illustrated in Figure 10. The overlap is similar in the omitted mesh structure.

The intermediate projector $\pi_b^* u$ is obtained from $\pi_b u$ after application of the liftings $\mathcal{L}_e[\pi_b u]_{\sigma}$ and $\mathcal{L}_e^2[\pi_b u]_{\sigma}$ on each of the submeshes $\tilde{\mathcal{M}}_{\sigma}^{\ell,ce_1} : \|$. While we have again $\pi_b^* u = \pi_b u$ on diagonal elements, the approximation $\pi_b^* u$ is non-conforming across edge-perpendicular faces of the subdiagonal elements given by

$$\mathcal{D}^{\ell'} : = \left\{ \begin{array}{l} \{ K''_1, K''_1, K''_3, K''_3, K'''_1, K'''_3, K''_5, K'''_5, K''_6, K'''_6 \}, \\
\{ K''_1, K''_1, K''_1 \}, \end{array} \right\}, \quad 2 \leq \ell' \leq \ell, \quad \ell' = 1,$$

(5.79)

employing notation as in Figure 11. As before, the definitions (5.69), (5.73) of the projector $\pi_b^{D,*} u$ can be generalized to the configuration in Figure 11, by using the subdiagonal elements in (5.79). This gives rise to a conforming approximation $\pi_b^{D,*} u$

![Fig. 11. Refinement toward three edges. Left: Outermost edge-patch blocks for $\sigma = 0.5$ and $\ell = 5$. Right: Interface between $\tilde{\mathcal{X}}_{e}^{\ell,ce} = \{ K'_1 \}$ and the first edge-patch block for $\sigma = 0.5$ and $\ell = 2$.](image-url)
over \( \bigcup_{i=1}^{3} M_{\sigma}^{ce} \). It is also continuous across the faces \( \int_{K_i, K_j}, \int_{K_i, K_j} \), \( \int_{K_i, K_j} \), and \( \int_{K_i, K_j} \) in Figure 11 (right). Since \( \pi_H u \) is identically zero on \( K_i, K_j \in \mathcal{T}_H \), cp. (4.10), there is no need for further liftings into \( \mathcal{T}_H \). Therefore, Proposition 5.4 remains valid for corner-edge patches which are geometrically refined along three edges \( e \in E \), which meet at a convex vertex \( c \).

Finally, we emphasize that the case of edges meeting at a reentrant vertex \( c \), i.e. a Fichera corner, can be dealt with by superposition of the preceding constructions on seven patches: one corner patch (cp. Section 5.3), three corner-edge patches with a single edge (cp. Figure 8) and three corner-edge patches with two edges meeting at \( c \) (cp. Figure 10). In view of Assumptions 3.1, 3.2, the lifting constructions given above are conforming across patch interfaces.

6. Acknowledgment

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC) and by the European Research Council (ERC) under grant AdG STAHDPDE 247277.

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