L13: Polar Coordinates

\[ x(R, \theta) = R \cos \theta \]
\[ y(R, \theta) = R \sin \theta \]
\[ R^2 = x^2 + y^2 \quad \theta = \arctan \left( \frac{y}{x} \right) \]

Ex: Determine the set of points described by \( R = 2 \cos \theta \).

\[ \text{Sol:} \quad R^2 = 2R \cos \theta. \]
\[ x^2 + y^2 = 2x \]
\[ (x-1)^2 + y^2 = 1 \quad \rightarrow \]
\[ D = \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 = 1\} \]

Ex: Describe in polar coordinates the set:
\[ D = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 2, x > 0, y \geq 0\} \]

\[ \text{Sol:} \quad D = \{(R, \theta) \in \mathbb{R}^+ \times [0, 2\pi] \mid 1 \leq R \leq 2, 0 \leq \theta \leq \frac{\pi}{2}\} \]
**Def:** A polar rectangle is a set $R$ of
the form $R = \{(r, \theta) : a \leq r \leq b, \alpha \leq \theta \leq \beta\}$

**Q:** How to compute $\iint_{R} f(x, y) \, dA$?

**Ans:** Same Riemann Sum Idea.

Note: Area of slice of pie

$\frac{\theta}{2\pi} \pi R^2 = \frac{\theta R^2}{2}$

So for a rectangle $R_{ij}$, we have

$\text{area}(R_{ij}) = \frac{1}{2} \Delta \theta \left( (R_{i+1, j+1})^2 - (R_{ij})^2 \right) = \frac{1}{2} \Delta \theta \left( \Delta R^2 + 2R\Delta R \right) = \frac{1}{2} \Delta \theta R \Delta R + \frac{1}{2} \Delta \theta \Delta R^2$. 

\[ \sum \]
So \( \text{area}(R) = \lim_{\Delta \theta, \Delta R \to 0} \sum_{i=1}^{m} \sum_{j=1}^{m} \Delta \theta \Delta R \left( R_i + \frac{1}{2} \Delta R \right) \)

but \( \lim_{\Delta R \to 0} \sum_{i=1}^{m} \Delta R \left( \frac{1}{2} \Delta R \right) = 0 \) as \( \sum_{i=1}^{m} \Delta R = R \).

so we can consider \( \text{area}(R) = \lim_{\Delta \theta, \Delta R \to 0} \sum_{i,j=1}^{m,m} \Delta \theta \Delta R R_i \).

**Def.: Polar Riemann Sum.**

\[ S_{m,m} = \sum_{i=1}^{m} \sum_{j=1}^{m} f(R_i \cos \theta_j, R_i \sin \theta_j) \cdot R_i \cdot \Delta R \Delta \theta. \]

and \( \int_{R} f(R \cos \theta, R \sin \theta) \, dA = \lim_{m,m \to \infty} S_{m,m} \).

Taking the limit above we obtain:

\[ \iint_{R} f(x,y) \, dA = \int_{a}^{b} \int_{a}^{b} f(R \cos \theta, R \sin \theta) \, R \, dR \, d\theta \]

Think of "\( dxdy = RdRd\theta \)"

Note \( \det \left( \begin{bmatrix} \frac{\partial x}{\partial R} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial R} & \frac{\partial y}{\partial \theta} \end{bmatrix} \right) = \det \left( \begin{bmatrix} \cos \theta & \sin \theta \\ -R \sin \theta & R \cos \theta \end{bmatrix} \right) = 0 \, R \)
Ex: evalu I = \iiint_R x^2 \, dA \quad R = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq r \leq 2, \quad 0 \leq \theta \leq \frac{\pi}{2}\}

\text{Sol.} \quad I = \int_{\frac{\pi}{2}}^{\pi} \int_{1}^{\sqrt{2}} r^2 \cos^2 \theta \, r \, dr \, d\theta

= \left[ \frac{r^4}{4} \right]_{1}^{\sqrt{2}} \int_{\frac{\pi}{2}}^{\pi} \cos^2 \theta \, d\theta

= \frac{15}{4} \int_{0}^{\frac{\pi}{2}} \frac{1 + \cos 2\theta}{2} \, d\theta

= \frac{15}{8} \left( \frac{\pi}{2} + \frac{1}{2} \sin(2\theta) \right) \bigg|_{0}^{\frac{\pi}{2}} = \frac{15\pi}{16}

\text{Remark: We can also integrate over Type I, II region.}

Ex. Find the value below the paraboloid 
Z = x^2 + y^2 and above the xy-plane and inside the cylinder (x-1)^2 + y^2 = 1.
Note that 
\((x-1)^2 + y^2 \leq 1\) 
\(0 \leq R \leq 2 \cos \theta\).

So \(D = \{(R, \theta) : \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}, 0 \leq R \leq 2 \cos \theta\}\).

So 
\[ V = \int \int_D (x^2 + y^2) \, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2 \cos \theta} R^2 \, RdR \, d\theta. \]

\[ = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{2 \cos \theta}{4} \right)^4 \, d\theta = \int_{0}^{\frac{\pi}{2}} \frac{8 \cos^4 \theta}{4} \, d\theta. \]

\[ \cos^4 \theta = \left( \frac{1 + \cos(2\theta)}{2} \right)^2 = \frac{1}{4} \left( 1 + 2 \cos 2\theta + \cos^2 2\theta \right) \]

\[ = \frac{1}{4} \left( 1 + 2 \cos 2\theta + \frac{1 + \cos(4\theta)}{2} \right) \]

\[ = \frac{3}{8} + \frac{4 \cos 2\theta}{8} + \frac{\cos 4\theta}{8}. \]

So 
\[ V = \int_{0}^{\frac{\pi}{2}} \left( 3 + 4 \cos 2\theta + \cos 4\theta \right) \, d\theta = \left[ 3\theta + 2 \sin 2\theta + \sin 4\theta \right]_{0}^{\frac{\pi}{2}} \]

\[ = 3 \left( \frac{\pi}{2} \right). \]
Ex: Bell curve.

\[ f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \]

Eval \[ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx \]

\{ problem: \( e^{-\frac{x^2}{2}} \) has no elementary primitive \}

Sol: \[ I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx \cdot \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \, dy. \]

\[ = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^2} e^{-\frac{(x^2+y^2)}{2}} \, dx \, dy \]

\[ = \frac{1}{\sqrt{\pi}} \int_{0}^{2\pi} \int_{0}^{\infty} e^{-\frac{R^2}{2}} \, R \, dR \, d\theta. \]

\[ = \int_{0}^{\infty} R e^{-\frac{R^2}{2}} \, dR = \int_{0}^{\infty} e^{-u} \, du = -e^{-u} \bigg|_{0}^{\infty} \]

\[ u = \frac{R^2}{2} \quad du = RdR \quad = 0 - (-1) = 1 \]

So \[ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx = 1 \]
Ex: Find the area of the region outside the circle of radius $R = \sqrt{2}$ centered at $(0,0)$ and inside the circle of radius 1 centered at $(0,1)$.

Sol:

Find intersection:

$\sqrt{(y-1)^2} = 1$

$\sqrt{y^2 - (y-1)^2} = 1$

$-1 + 2y = 1 \implies y = 1, x = \pm 1$

With FIRST YEAR CALCULUS

$A = \int_{-1}^{1} [(1+\sqrt{1-x^2}) - \sqrt{2-x^2}] dx$

$= 2 + \frac{\pi}{2} - 2 \int_{0}^{1} \sqrt{2-x^2} \, dx$

$= 2 + \frac{\pi}{2} - 2 \left[ \int_{0}^{\pi/4} \sqrt{2-2\cos^2 \theta} \cdot \sqrt{2} \cos \theta \, d\theta \right]$

$= 2 + \frac{\pi}{2} - 4 \int_{0}^{\pi/4} \cos^2 \theta \, d\theta$

$= 2 + \frac{\pi}{2} - 2 \left[ \frac{\theta + \sin 2\theta}{2} \right]_{0}^{\pi/4}$

$= 2 + \frac{\pi}{2} - \frac{\pi}{2} - 1 = 1$

High School:

$A = \frac{1}{4}(2\pi)^2 - \frac{1}{2}(2\pi^2)$

$+ \frac{1}{2}(\pi^2 - (\pi/2)^2)$

$- 1$. 

$A = \frac{3\pi^2}{4}$. 

$\int_{\pi/4}^{3\pi/4} \int_{\sqrt{1-y_1}}^{\sqrt{2-y_1}} R \, dR \, d\theta$

$= \int_{\pi/4}^{3\pi/4} \frac{3\pi^2}{8} \left( \frac{\sin 2\theta}{2} - \frac{1}{2} \right) \, d\theta$

$= \int_{\pi/4}^{3\pi/4} \frac{3\pi^2}{8} \left( 1 - \cos(2\theta) - 1 \right) \, d\theta$

$= -\frac{\sin (2\theta)}{2} \bigg|_{\pi/4}^{3\pi/4}$

$= \frac{1}{2} \left[ \sin(\pi/2) - \sin(3\pi/2) \right]$

$= \frac{1}{2} (1 - (-1)) = 1$.