Suppose \( x = g(t) \) and \( y = h(t) \) are differentiable \( x \), \( y \).
and \( z = f(x, y) = f(g(t), h(t)) \) is differentiable.

Let \( \Delta x = g(t + \Delta t) - g(t) \) \( \Delta y = h(t + \Delta t) - h(t) \).

Then: \( \Delta z = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \).

Dividing by \( \Delta t \) we get:

\[
\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x} \frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y} \frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}.
\]

Take \( \Delta t \to 0 \), \( \varepsilon_1, \varepsilon_2 \to 0 \) as \( \Delta t \to 0 \) become \( \Delta x, \Delta y \to 0 \) as \( \Delta t \to 0 \).

We have that:

\[
\frac{\Delta x}{\Delta t} = \frac{g(t + \Delta t) - g(t)}{\Delta t} \to g'(t) = \frac{dg}{dt}(t)
\]

\[
\frac{\Delta y}{\Delta t} = \frac{h(t + \Delta t) - h(t)}{\Delta t} \to h'(t) = \frac{dh}{dt}(t)
\]

We get:

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]

**Chain Rule, 1st Version.**
Informally: \[ \frac{dz}{dt} = \frac{dz}{dx} \frac{dx}{dt} + \frac{dz}{dy} \frac{dy}{dt} \]

"and divide by \( dt \)"

ok, so, what about more variables.

\[ x = g(s,t) \quad y = h(s,t) \]

Sol: Fix one variable. We are in the previous case, so...

\[
\frac{dz}{ds} = \frac{\partial z}{\partial x} \frac{dx}{ds} + \frac{\partial z}{\partial y} \frac{dy}{ds}
\]

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]

Chain rule

2nd version,

in matrix form.

\[
\begin{bmatrix}
    \frac{\partial z}{\partial s} \\
    \frac{\partial z}{\partial t}
\end{bmatrix}
= \begin{bmatrix}
    \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\
    \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t}
\end{bmatrix}
\begin{bmatrix}
    \frac{\partial z}{\partial x} \\
    \frac{\partial z}{\partial y}
\end{bmatrix}
\]

Derivatives are partial.
Example 1: Suppose \( z = f(x, y) \) and \( x(R, \theta) = R \cos \theta, \quad y(R, \theta) = R \sin \theta \). 

\[
\frac{\partial z}{\partial R} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial R} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial R} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta.
\]

\[
\frac{\partial z}{\partial \theta} = -R \sin \theta \frac{\partial z}{\partial x} + R \cos \theta \frac{\partial z}{\partial y}.
\]

For \( z = x^2 + y^2 \), we get:

\[
\frac{\partial z}{\partial R} = 2x \cos \theta + 2y \sin \theta = 2R (\cos^2 \theta + \sin^2 \theta) = 2R
\]

\[
\frac{\partial z}{\partial \theta} = -R \sin \theta (2x) + R \cos \theta (2y) = 2R (\sin \theta \cos \theta + \sin \theta \cos \theta) = 0,
\]

which we can verify directly.

\[
z = x^2 + y^2 = R^2 \quad \text{so} \quad \frac{\partial z}{\partial R} = 2R \quad \frac{\partial z}{\partial \theta} = 0.
\]

Example 1: Let \( w = \ln \left( \sqrt{x^2 + y^2 + z^2} \right) \). \( x = \sin(t) \), \( y = \cos(t) \), \( z = \tan(t) \).

Compute \( \frac{dw}{dt} \).

Solution:

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}.
\]

\[
\frac{\partial w}{\partial x} = \frac{x}{x^2 + y^2 + z^2}, \quad \frac{\partial w}{\partial y} = \frac{y}{x^2 + y^2 + z^2}, \quad \frac{\partial w}{\partial z} = \frac{z}{x^2 + y^2 + z^2}.
\]
also \(\frac{dx}{dt} = \cos(t)\), \(\frac{dy}{dt} = -\sin(t)\), \(\frac{dz}{dt} = \sec^2(t) = \frac{1}{\cos^2(t)}\).

so \(\frac{dw}{dt} = \frac{1}{x^2 + y^2 + z^2}\left( x \cos(t) - y \sin(t) + z \cdot \frac{1}{\cos^2(t)} \right)\).

\(x^2 + y^2 + z^2 = \cos^2(t) + \sin^2(t) + \tan^2(t) = 1 + \tan^2(t) = \frac{1}{\cos^2(t)}\).

so \(\frac{dw}{dt} = \cos^2(t) \left[ \cos^2(t) \sin(t) - \cos(t) \sin(t) + \frac{\tan(t)}{\cos^2(t)} \right]\).

\[\frac{dw}{dt} = \tan(t)\]

**Exercise**: CHECK DIRECTLY.

**Ex1** CONSIDER A CONE.

THE RADIUS IS INCREASING AT 1.8 cm/s, AND THE HEIGHT IS DECREASING AT 2.5 cm/s. AT WHAT RATE DOES THE VOLUME CHANGE?

(SOL) \(V = \frac{1}{3} \pi R^2 H\).

\[\frac{dV}{dt} = \frac{2}{3} \pi R^2 \frac{dH}{dt} + \frac{1}{3} \pi R^2 \frac{dR}{dt}\]

\[= \frac{2}{3} \pi R^2 \frac{dH}{dt} + \frac{1}{3} \pi R^2 \frac{dR}{dt}\]

\[= \frac{2}{3} \pi \cdot (120)\cdot (140)\cdot 1.8 + \frac{1}{3} \pi \cdot (120)^2 \cdot \frac{2}{3}\]

\[\approx 26,635 \text{ cm}^3/\text{s}\]
**Implicit Differentiation**

Consider \( x^3 - y^4z + e^{x^2} = 0 \)

Assume \( z = f(x,y) \) implicitly.

More generally, we can have \( F(x,y,z) = 0 \).

Which we assume is \( F(x,y,f(x,y)) = 0 \) implicitly.

**Is this always possible?** A: "Implicit Function Thm".

**Q:** How to compute \( \frac{\partial z}{\partial x} \) or \( \frac{\partial z}{\partial y} \)? \( A: \) Chain Rule.

\[
0 = \frac{\partial}{\partial x} \left( x^3 - y^4z + e^{x^2} \right) = 3x^2 - \frac{\partial F}{\partial x} \frac{\partial x}{\partial x} - 4y^3z \frac{\partial x}{\partial x} + 2xe^{x^2} \frac{\partial x}{\partial x}.
\]

\[
0 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x}.
\]

So \( \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \) and \( \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \).

**Example** \( x^3 - y^4z + e^{x^2} = 0 \).

\[
\frac{\partial z}{\partial x} = \frac{-3x^2 + 2xe^{x^2} - y^4}{2ye^{x^2} - y^4}, \quad \frac{\partial z}{\partial y} = \frac{-4y^3z}{2ye^{x^2} - y^4}.
\]

At \( (0,1,1) \) the tangent plane is: \( Z = 1 + Z_x(0,1)(x-0) + Z_y(0,1)(y-1) \)

\( Z = 1 + (1)(x-0) + (-4)(y-1) \implies \boxed{Z = x - 4y + 5} \).