Lecture 4: We saw some simple surfaces in $\mathbb{R}^3$.

Consider $z = y^2 - x^2$. How does this look like?

We can use "trace curves" to visualize it:

\[ z = 0 \]

Hyperbola.

\[ z = -1 \]

Hyperbola.

\[ x = 0 \]

Parabola.

\[ y = 0 \]

Parabola.

We get:

Hyperbolic Paraboloid.

(see 3D Sym)
A nice way to visualize is with a contour plot. (Level curves for $z$).

Ex: $x^2 + y^2 + z^2 = 1$

Function of several variables ($z = f(x,y)$).

$f: D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$.

A function $f$ with domain $D \subseteq \mathbb{R}^n$ assigns each $(x_1, ..., x_n) \in D$ a real number $f(x_1, x_2, ..., x_n)$. 
(a) \( f(x,y) = \sqrt{1-x^2-y^2} \) \( D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\} \)

(b) \( g(x,y) = \text{Im}(x+y) \) \( D = \{(x,y) \in \mathbb{R}^2 : x+y > 0\} \)

(c) \( h(x,y) = \frac{1}{x^2+y^2} \) \( D = \mathbb{R}^2 \setminus \{(0,0)\} \)

(d) \( k(x,y) = e^{x^2+y} + \sin(y^4+3x) \) \( D = \mathbb{R}^2 \)

Graphs of functions:

\( f: D = \mathbb{R}^2 \) \( \longrightarrow \mathbb{R} \)

Graph of \( f(x) = y \) a curve.

\( f(x,y) = z \) The graph is a surface.
Ex: \[ f(x, y) = \sqrt{1 - x^2 - y^2} \]

A good idea to visualize a graph is to use the contour plot.

Q: How to “graph” a function \( f: \mathbb{R}^3 \rightarrow \mathbb{R} \)? “Need 4D"

A: Use contour surfaces. Use time simulation w/o computer.
Limits and continuity.

Recall $y = f(x)$.

$\lim_{x \to a} f(x) = L$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in (a - \delta, a + \delta)$ and $x \neq a$.

Remark: $\lim_{x \to a^-} f(x) = L \iff \lim_{x \to a^+} f(x) = L$.

Example: $\lim_{x \to a} \sin \left( \frac{1}{x} \right)$ does not exist.

$\lim_{x \to 0} x \sin \left( \frac{1}{x} \right) = 0$.

Continuous but not differentiable.

* NOT EXAMINABLE *.
On two or more variables things are more complicated. Why? May directions to approach $(a,b)$. 

\[ \lim_{{(x,y) \to (0,0)}} \frac{x^2}{x^2 + y^2} = ? \]

Let's take it along the line $y = mx$.

\[ \lim_{{(x, mx) \to (0,0)}} \frac{x \cdot mx}{x^2 + m^2x^2} = \lim_{{(x, mx) \to (0,0)}} \frac{mx}{1 + m^2} = \frac{M}{1 + m^2}. \]

but this depends on $M$.

(See computer simulation)

So, what is the def off limit here?

Let $f: \mathbb{D} \subseteq \mathbb{R}^n \to \mathbb{R}$.

We say $\lim_{{\vec{w} \to \vec{w}_0}} f(\vec{w}) = L$ if

\[ \forall \varepsilon > 0 \quad \exists \delta > 0 \quad 0 < \| \vec{w} - \vec{w}_0 \| < \delta \implies \| f(\vec{w}) - L \| < \varepsilon. \]

and $\vec{w} \in \mathbb{D}$.

Ex: $f: \mathbb{D} \subseteq \mathbb{R}^2 \to \mathbb{R}$. 

\[ \text{CONTAINED IN} \]

\[ \Delta \]

$\varepsilon$
Q: Would it be OK if the limit along every line \( y = mx \) existed and were the same?

A: \( \text{NO: } D = (\mathbb{R}^+)^2 = \{ (x,y) \in \mathbb{R}^2 : x > 0 \land y > 0 \} \)

\[
f(x,y) = e^{-\frac{x}{y(x^2 + y^2)}}.
\]

Let \( g = mx \)

\[
\lim_{(x, mx) \to (0,0)} e^{-\frac{x}{mx(x^2 + m^2x^2)}} = \lim_{x \to 0} e^{-\frac{1}{m(x^2 + m^2x^2)}} = 0
\]

so it should be 0, right? \( \text{NO} \)

\[\text{Computer Says}\]

Solve \( f(x,y) = \frac{1}{2} \), give a curve for which it does not work.

BUT \( \exists \) IT IS OK to converge in EVERY curve i.e.: if for every \( \gamma : [0,1] \to \mathbb{R}^2 \) such that \( \gamma(1) = (a,b) \) we have \( \forall t < 1 \gamma(t) \neq (a,b) \) and \( \gamma(t) \in D \),

\[
\lim_{t \to 1} \gamma(t) = L
\]

\[
\text{then } \lim_{(x,y) \to (a,b)} f(x,y) = L
\]