A dynamical system is a set $X$ endowed with a topology and a group $G$ acting over $X$ by continuous transformations. In many cases arising naturally, such as the case of axiom A diffeomorphisms [Bow70], these systems can be represented by shift spaces, that is, as closed and shift-invariant sets of colorings of the group $G$ by a finite alphabet $A$ endowed with the shift action. This representation is of special interest as it encodes the dynamical properties of the original action in the set while turning the action into a translation.

Shift spaces, also known as subshifts, are also quite interesting as they are both dynamical objects –closed and shift-invariant sets– and combinatorial ones –sets of colorings defined by forbidding the appearance of a list of patterns. This equivalence comes from the fact that closed sets in $A^G$ are complements of unions of cylinders and therefore if we denote for a finite $F \subseteq G$ and a pattern $p \in A^F$ the cylinder $[p] = \{ x \in A^G \mid \forall g \in F, x_g = p_g \}$ then, for a list of patterns $\mathcal{F}$

$$X = A^G \setminus \bigcup_{g \in G, p \in \mathcal{F}} g \cdot [p]$$

is closed and shift invariant.

A particular interesting case arises when the subshift is of finite type (SFT), that is, when the set of forbidden patterns $\mathcal{F}$ is finite. If the group $G$ has a decidable or at least recursively enumerable word problem it allows these objects to be described by Turing machines and algorithmic questions can be asked and related to dynamical ones. The interplay between these two worlds has proven to be particularly fruitful. A seminal example is the characterization of the numbers which can be obtained as topological entropies of $\mathbb{Z}^2$-SFTs as right recursively enumerable numbers [HM10]. This result was achieved by techniques mixing elements of both areas. Specifically, by coding Turing machines inside of SFTs.

The main goal of this research project is to further develop this interplay in various directions. I will begin by briefly presenting some of the previous projects I have participated in during my thesis which fall into the topic and then proceed to detail my research project.

**Previous work**

Most of the work described in this section has been done in the context of my thesis, which revolved around the study of dynamical and algorithmic properties of subshifts in groups. The articles described here all explore this subject from different angles.

- In **A notion of effectiveness for subshifts on finitely generated groups** [ABS17] written in collaboration with Nathalie Aubrun et Mathieu Sablik and published in *Theoretical computer science*, we extended the standard notion of effective subshift for a $\mathbb{Z}$ action to the case of arbitrary finitely generated groups $G$. The classical way of defining an effective subshift $X \subseteq A^\mathbb{Z}$ is to ask for the existence of a recursively enumerable list of forbidden words. In this article we consider the obvious extension to arbitrary groups by means of codings of patterns and study their properties. We found several negative properties over this class of subshifts as soon as the word problem of the group $G$ becomes less tractable and thus introduced the notion of $G$-effectiveness which adds the power of an oracle to the word problem of the group.

This notion, even though being misleadingly artificial, admits a natural characterization by modified Turing machines which, instead of working over a tape, replaces it by a Cayley graph of the group and move
using a fixed set of generators. As a consequence of this study we found the first examples of finitely generated groups such that the subshift $X \subseteq \{0,1\}^G$ consisting of all configurations allowing at most one appearance of the symbol 1 is not a sofic subshift, thus answering a question posed by Dahmani and Yaman from one of their articles [DY08]. We also constructed a simulation theorem in the spirit of the work of [AS13] which works in any finitely generated group under some conditions. This construction strongly uses these modified Turing machines and Delone sets.

- In the article Realization of aperiodic subshifts and densities in groups [ABT15] written with Nathalie Aubrun and Stéphan Thomassé and currently accepted for publication in Groups, Geometry and Dynamics, we construct a new technique based on Lovász local lemma which gives a strong tool to prove that a subshift defined by a set of forbidden patterns is non-empty. As an application of this tool we give a new proof of the existence of non-empty strongly aperiodic subshifts over the alphabet $\{0,1\}$ for any countable group $G$. That is, non-empty subshifts for which the shift action of $G$ is free. While this result was already known from [GJS09] our techniques allow us to make sure the construction can be made effective as long as $G$ is finitely generated and its word problem is decidable. We also used similar techniques to prove that in amenable and finitely generated groups there always exist subshifts $X \subseteq \{0,1\}^G$ such that for any configuration $x \in X$ and any Følner sequence in $G$ the density of 1s converges to a fixed arbitrary value $\alpha \in [0,1]$.

- In the article A generalization of the simulation theorem for semidirect products [BS16b] with Mathieu Sablik and recently accepted in Ergodic Theory and Dynamical Systems, we study an extension to Hochman’s simulation theorem [Hoc09] for general groups. We proved that any $G$-action of a finitely generated group over a Cantor set which can be in a specific way encoded by a Turing machine can be realized as a factor of a subaction of a $\mathbb{Z}^2 \rtimes G$-subshift of finite type. As a corollary of this result and the construction in [ABT15] we prove that every group of the form $\mathbb{Z}^d \rtimes G$ admits non-empty strongly aperiodic subshifts of finite type as long as $G$ has decidable word problem, is finitely generated and $d \geq 2$.

- In The domino problem for self-similar structures [BS16a] written in collaboration with Mathieu Sablik and accepted in the conference CIE 2016, we study the decidability of the domino problem—i.e., the problem of whether the set of colorings of a structure defined by a finite number of forbidden patterns is empty—in the context of fractals. More specifically, we consider subsets of $\mathbb{Z}^d$ which are generated by substitutions mimicking finite subdivision rules. We studied these objects as intermediate instances of the domino problem of $\mathbb{Z}$ which is decidable and the domino problem of $\mathbb{Z}^2$ which is not. We found a class of substitutions for which the problem is decidable, notably including the Sierpiński triangle, and one for which is not, including the Sierpiński carpet. We also described an intermediate class for which the result remains open.

- The group of reversible Turing machines [BKS16] written with Jarkko Kari and Ville Salo, accepted in the conference AUTOMATA 2016, is currently being extended into a long journal version. Here we study reversible Turing machines as objects which form a group under composition. This object turns out to be a group which contains the countable topological fullgroup of the full shift $\{\sigma\}$, but that it is different from $\text{Aut}(\mathbb{Z})$, in particular it is not residually finite. We study the different group theoretical properties of this object and as a result of our study, we get the undecidability of the torsion problem—the computational problem of deciding whether there exists a power of a word which represents the identity—for a finitely generated subgroup of both $\text{Aut}(\mathbb{Z})$ and the countable topological fullgroup of a full shift in $\mathbb{Z}^d$ for $d \geq 2$. 
Research project: short term

In this section I mostly describe ongoing work and what are my expectations in the short term.

- **The DLR theorem** Classical theorems in thermodynamical formalisms by Dobrushin, and Lanford and Ruelle [LR69] give conditions that ensure the equivalence of the equilibrium measures of a dynamical system with its Gibbs measures. Several extensions of these results have been studied in the literature, for instance [Sep95, Rue04, Mey12]. In our project we aim to generalize these theorems simultaneously in many aspects: we consider actions of countable amenable groups, general interactions, relative versions and we weaken some of the conditions needed to obtain the results. We also envision several applications of our generalized theorem. This is ongoing work with Ricardo Gómez Aíza, Brian Marcus and Siamak Taati.

- **The domino problem for word hyperbolic groups** In the context of a group $G$, the domino problem for $G$ consists on deciding whether given a description of a finite set of forbidden patterns, the $G$-SFT defined by them is non-empty. There is no full characterization of which groups admit a decidable domino problem and the current standing conjecture asserts that they are exactly the virtually free groups. Our project aims to prove that this conjecture holds in the context of word hyperbolic groups. This is ongoing work with Nathalie Aubrun and Étienne Moutot.

- **A hierarchy of systems with TCPE** A result by Pavlov [Pav17] gives a necessary condition for a $\mathbb{Z}^d$-subshift of finite type admitting a fully supported measure to have Topologically Completely Positive Entropy (TCPE), that is, satisfy that every non-trivial dynamical factor has positive entropy. Pavlov also asks whether said condition is in fact sufficient. In this ongoing work with Felipe García Ramos we construct a counterexample. Furthermore, we construct a hierarchy of properties ensuring TCPE and show strict separation among these classes.

- **Strongly aperiodic SFTs in torsion groups** This is an extension of the project done with Mathieu Sablik in [BS16b]. Using the technique of translation-like actions, I managed to prove a similar theorem in the case of triple products of finitely generated groups with decidable word problem. As a corollary of this, I managed to show that every finitely generated branch group with decidable word problem admits a strongly aperiodic SFT, in particular, this applies to the Grigorchuk group [Gri84]. This work is currently on arXiv [Bar17] and submitted for publication.

- **A dynamical proof of Higman’s theorem** Higman’s theorem asserts that every recursively presented group embeds into a finitely presented group. This result has its natural analogue in the case of dynamical systems through the works of Hochman [Hoc09], Aubrun and Sablik [AS13] and Durand, Romashenko and Shen [DRS10]. I aim to use the latter techniques to recover the classical Higman’s theorem from purely dynamical arguments.

- **The group of reversible Turing machines** A long version of the article [BKS16] is currently under production. This is ongoing work with Jarkko Kari and Ville Salo.

Research project: long term

**Axis 1: Entropy, sofic groups and thermodynamical formalism**

Ergodic theory has classically been studied in the setting of a topological space endowed with a single transformation, and has since been generalized in the invertible case to allow for arbitrary countable amenable group actions, see for instance [OW80, Lin01]. In the recent years, an important component of this theory, more precisely entropy theory, has been generalized to an even wider setting: sofic groups [Wei00]. This generalization first introduced by Bowen [Bow10] has received much acclaim and gives a suitable setting for entropy theory to go beyond the scope of amenable group actions.

In this setting, one might expect that entropy theory in non-amenable groups should follow some characteristics of classical entropy theory. It is known that $\mathbb{Z}$-SFTs can only attain entropies which are logarithms of Perron numbers [LM95] while $\mathbb{Z}^2$-SFTs can realize any right recursively enumerable number [HM10]. However, there is still no characterization of the entropies of sofic approximations of SFTs defined on free groups. The main obstruction to this is the fact even though sofic entropy is well defined, the explicit computation of even the
simplest examples remains very complicated.

From another point of view, in the purely entropic case, the classical DLR theorem gives conditions under which the measures of maximal entropy coincide with the Gibbs measures of a system. These theorems can be generalized to the case of countable amenable group actions. It is the goal of this project to see up to which point this holds in the non-amenable case. While the computation of sofic entropy is complicated, the definition of a Gibbs measure through specifications is pretty much unchanged if the group is non-amenable, and thus this might give a simpler way to compute the sofic entropy in some cases, thus giving us some insight on sofic group actions.

**Axis 2: The domino problem for subshifts in groups**

Given a description of a finite set of forbidden patterns \( F \), a natural question is to ask whether the subshift of finite type \( X_F \) they define is non-empty. This is a more general version of the classical Domino Problem in which a finite set of square tiles with colored edges is given and the question is whether they can tile the plane in such a way that adjacent tiles carry the same color over their adjacent edge. This problem is known to be undecidable [Ber66, Rob71] while its analogous version in \( \mathbb{Z} \) is decidable. One might wonder for which finitely generated groups the domino problem is decidable. So far, it is known that virtually free groups have decidable domino problem [MS83, MS85] and as there are no examples outside this class it is conjectured these are the only ones. This conjecture is known to hold up to the class of polycyclic groups [Jea15] and thus the next logical step is to see if it holds in solvable groups.

Amongst this class, a fairly well understood example is the Lamplighter group \( \mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z} \) whose Cayley graph admits a nice representation by Diestel-Leader graphs [Woe04]. Surprisingly, the decidability of the domino problem for the Lamplighter group is yet unknown, furthermore, it is even unknown if the origin-fixed domino problem is decidable, that is, if there exists a Turing machine which can decide whether \( X_F \cap [a] = \emptyset \) for some symbol \( a \in A \). I intend to study this problem and if some result is found, to extend it as far as possible. My current work in this subject is exploring both the solvable and the word-hyperbolic case.

**Axis 3: Simulation theorems : free actions and entropy**

Simulation theorems following the work of Hochman [Hoc09, AS13, DRS10] and my recent contribution [BS16b] are interesting as they allow to extend some of the properties of effective subshifts onto subshifts of finite type. Namely, it is easier to construct an example of an effective subshift which satisfies a property than it is to do the same with a subshift of finite type.

On the one hand I intend to further investigate if one can obtain sharper or more general simulation theorems. While Hochman’s theorem simulates any effective \( \mathbb{Z} \)-action over a zero-dimensional set with the cost of adding a product with \( \mathbb{Z}^2 \), the theorems on [AS13, DRS10] realize only expansive actions but only add \( \mathbb{Z} \) as a product. The recent theorem I proved with Sablik [BS16b] and the current extension I proved in the case of torsion groups considerably extend the framework of Hochman, but do not seem to be optimal in the expansive case. My goal is to find out whether it is true that every expansive effectively closed \( G \)-action of a finitely generated group over a zero-dimensional set can be in fact obtained as the projective subdynamics of a \( G \times \mathbb{Z} \) or more generally, a \( G \times H \) sofic subshift for some f.g group \( H \).

On the other hand, simulation theorems are powerful tools to produce examples of strongly aperiodic subshifts of finite type, that is, subshifts where the shift acts freely. An illustrative example is the construction of the first strongly aperiodic SFT in the Grigorchuk group [Bar17]. This kind of theorem might also be useful in the quest to classify the entropies attainable by SFTs in general amenable groups. I intend to study if the current and possibly new simulation theorems can be applied or modified to satisfy those goals.
Axis 4: Automorphism groups of subshifts

The automorphism group $\text{Aut}(X)$ of a subshift $X \subset A^G$ consists of all shift-commuting homeomorphisms of $X$. It is a dynamical invariant in the sense that conjugate subshifts must have isomorphic automorphism groups but the converse is not true. This object has turned out to be quite difficult to handle, it is still open whether $\text{Aut}((0,1)^\mathbb{Z}) \cong \text{Aut}((0,1,2)^\mathbb{Z})$ even if it is known that both groups embed into each other [MBR88]. Recently, there has been an explosion of results on automorphism groups in the case where $X$ is not too complicated. For instance, in the case where $X$ has low word complexity there are the results [CK15, DDMP16] and in the case where $X$ is countable and has low Cantor-Bendixon rank there is the yet unpublished work of Salo and Schraudner. There has also been new advances which show some obstructions over which groups can be realized as an automorphism group of a subshift [CFKP16].

A particularly intriguing case is that of algebraic subshifts. Some examples have relatively small automorphism groups. For instance, in the case where $X$ has low word complexity there are the results [CK15, DDMP16] and in the case where $X$ is countable and has low Cantor-Bendixon rank there is the yet unpublished work of Salo and Schraudner. There has also been new advances which show some obstructions over which groups can be realized as an automorphism group of a subshift [CFKP16].

A particularly intriguing case is that of algebraic subshifts. Some examples have relatively small automorphism groups. For instance, the three-dot subshift $X_{Led} = \{ x \in \{0, 1\}^\mathbb{Z}^2 \mid \forall (i,j) \in \mathbb{Z}^2, x(i,j) + x(i+1,j) + x(i,j+1) = 0 \mod 2 \}$ satisfies $\text{Aut}(X_{Led}) \cong \mathbb{Z}^2$ while other examples, as the one satisfying $x(i,j) + x(i+1,j) + x(i,j+1) + x(i+1,j+1) = 0$, contain the automorphism group of a full $\mathbb{Z}$-shift. Algebraic dynamical systems have been studied thoroughly by Schdmit [Sch95] and nowadays there are still many open questions concerning them. In connection with this, one can define a notion of determinism in these subshifts which is a natural generalization of the notion used in [Kar92] to show the undecidability of nilpotence for cellular automata. This notion along with its natural dual allow to define a basis for algebraic subshifts which makes them amenable to analytic methods. My goal in this respect is to study up to which point this tool can be used to yield results about the automorphism group and to what level the determinism can be described by Turing machines.

In a different angle, I am also interested to study in what ways it is possible to take a subshift $X$ over a group $G$ and construct a new subshift $Y$ over a subgroup $H \leq G$ whose automorphism group is close to that of $X$. There are some obvious tools such as the projective subdynamics (see for instance [AS13, DRS10]) which do not work well in this respect and even if there were some technique to do this, it would not be possible to obtain any group as corollary from the obstructions of [CFKP16]. Nevertheless it may happen that some constrained class of subshifts does have some stability properties. I aim to study this problem in hopes of finding new tools to produce examples of automorphism groups. In particular, I am interested in finding out whether the discrete Heisenberg group might be obtained as an automorphism group of a $\mathbb{Z}$-subshift. This group is an excellent candidate as all finitely generated abelian groups have already been realized and it is the simplest nilpotent example not yet done.

References


