Last time: We looked at the graphs of the functions \( f(x) = 2^x \), \( g(x) = 3^x \), \( h(x) = e^x \). For any \( \alpha > 1 \), the graph of \( f(x) = \alpha^x \) looks like the following shape:

\[
\begin{align*}
\text{Graph of } f(x) = \alpha^x & \\
& \text{with } \alpha > 1
\end{align*}
\]

What about the graph of \( f(x) = \left(\frac{1}{\alpha}\right)^x \)? It turns out the graph of \( f(x) = \alpha^x \) where \( 0 < \alpha < 1 \) looks something like

\[
\begin{align*}
\text{Graph of } f(x) = \alpha^x & \\
& \text{with } 0 < \alpha < 1
\end{align*}
\]

Let's think about the case \( \alpha = \frac{1}{2} \). The function \( f(x) = \left(\frac{1}{2}\right)^x \) is always positive, and when \( x \) gets bigger and bigger (more positive), \( f(x) \) gets smaller and smaller (closer to 0). Think about \( x = 100 \). The output is \( f(100) = \left(\frac{1}{2}\right)^{100} = \frac{1}{2^{100}} \) which is a small number.

If \( x \) gets more negative (say \( x = -100 \)), \( f(x) \) gets big. For example, \( f(-100) = \left(\frac{1}{2}\right)^{-100} = 2^{100} \) which is huge.
Logarithms

$2^3 = 8 \implies \log_2 8 = 3$.
So, $\log_b x$ measures the exponent which $b$ (the base) needs to be raised in order to get $x$.

$5^2 = 25 \implies \log_5 25 = 2$.

$2^{-1} = \frac{1}{2} \implies \log_2 \left( \frac{1}{2} \right) = -1$.

"the base"

$\log_b x =$ exponent for which $b \leq$ needs to be raised in order to obtain $x$.

More tricky example: What is $\log_{25} 5$?
In other words, we are looking for the exponent $? \leq$ for which $25^? = 5$. The answer is that $? = \frac{1}{2}$, since $25^{\frac{1}{2}} = \sqrt{25} = 5$. So,

$\log_{25} 5 = \frac{1}{2}$.

$\log_{49} 49 = 2$ because $7^2 = 49$.

$\log_7 7^2 = 2$. Similarly, $\log_7 7^3 = 3$.

We see that $\log_7 7^n = n$ because $n$ is the exponent which we need to raise 7 to get $7^n$.

There was nothing special about 7. In fact,

$\log_b b^x = x$ for any base $b$, and for any $x$.

We assume $b > 0$ and $b \neq 1$. 
Example: \(5^{\log_5(25)} = 5^2 = 25\).

What should be \(5^{\log_5(8)}\)? We can guess that the answer should be 8. But why?

Answer: \(\log_5(8)\) is the exponent you need to raise 5 in order to get 8. Let's do it: let's raise 5 to this exponent \(\log_5(8)\)...

We get \(5^{\log_5(8)} = 8\) because \(\log_5(8)\) was, by definition, the right exponent to raise 5 to get 8.

This rule works in general:

\[ b^{\log_b x} = x \]

The explanation is the same as above: We know that \(\log_b x\) is the exponent we need to raise \(b\) in order to get \(x\).

So, \(b^{\log_b x} = x\).

More logarithm Rules

\(\log_b 1 = 0\) because \(b^0 = 1\).

\(\log_b xy = \log_b x + \log_b y\)

\(\log_b (x^k) = k \log_b x\).

Example: \(\log_3 7 + \log_3 5 = \log_3 (7 \cdot 5) = \log_3 (35)\)

\(\log_3 (2^{100}) = 100 \log_3 2\).