Exponential Functions

Review of exponent rules:

\[ 2^3 \cdot 2^5 = (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) = \overset{8 \text{ copies of } 2}{\overbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}} = a^8 \]

So \( a^3 \cdot a^5 = a^{3+5} = a^8 \). In general, \( a^m \cdot a^n = a^{m+n} \) for any whole numbers \( m \) and \( n \). There was nothing special about \( 2 \). The same rule works \( a^m \cdot a^n = a^{m+n} \) for any \( a \). How about \( (2^3)^3 \)?

\[ (2^3)^3 = (2^3)(2^3)(2^3) = 2^{3+3+3} = 2^9 = 512 \]

In general, \( (a^m)^n = a^{mn} \). The rule works for any \( a \) (instead of \( 2 \)), so \( (a^m)^n = a^{mn} \).

How shall we define \( f(x) = 2^x \)?

We know what \( 2^x \) means when \( x \) is a whole number: \( 2^m = 2 \cdot 2 \cdot \ldots \cdot 2 \), \( \frac{m \text{ times}}{m} \). What \( 2^{-m} \)?

We define \( 2^{-m} = \frac{1}{2^m} \) (this is consistent with \( 2 \cdot 2 = \frac{2}{2} = 2^0 = 1 \)).

This means we know how to evaluate the function \( f(x) = 2^x \) when \( x \) is an integer (Recall that integers are \( \ldots -5, -4, -3, -2, 1, 0, 1, 2, 3, 4, 5, \ldots \)).

What about if \( x = \frac{m}{n} \) is a rational number?

Then we can define \( 2^{\frac{m}{n}} = (2^m)^\frac{1}{n} = \sqrt[n]{2^m} \).

If \( x \) is not a rational number (think about \( x = \pi \)), then we can first approximate \( x \) with rational numbers, and then apply \( f(x) = 2^x \) to these approximations...
For example, \( f(x) = 2^x \) and say we want to understand \( f(\pi) \). Since \( \pi = 3.1415\ldots \) we can evaluate \( 2^3, 2^3.1, 2^{3.14}, 2^{3.141}, 2^{3.1415}, \ldots \) and in the limit (we will soon talk about this), the values above will tend to a certain fixed quantity which is what we call \( 2^\pi \).

In general, let \( a > 0 \). We can define the function \( f(x) = a^x \). The case \( a = 1 \) is boring (since \( 1^x = 1 \) for every \( x \)), so we will assume from now on that \( a \neq 1 \).

Let's graph \( f(x) = a^x \) for various values of \( a \).

If \( a > 1 \), the graphs look like this.

General shape: \( a > 1 \)

Never touches \( x \)-axis.

If \( a < 1 \), the graphs look like this.

General shape: \( a < 1 \)

Never touches \( x \)-axis.
### Laws of Exponents

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<tr>
<th>Law</th>
<th>Example</th>
<th>Comments/Justifications</th>
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<tbody>
<tr>
<td>$x^0 = 1$</td>
<td>$8^0 = 1$</td>
<td>$x^{-1} = \frac{1}{x}$, so $x^0 \cdot x^{-1} = x^0 \cdot \frac{1}{x} = x \cdot (\frac{1}{x}) = 1$.</td>
</tr>
<tr>
<td>$x^1 = x$</td>
<td>$7^1 = 7$</td>
<td>Any number raised to 1 is itself.</td>
</tr>
<tr>
<td>$x^{-1} = \frac{1}{x}$</td>
<td>$\frac{1}{5} = \frac{1}{5}$</td>
<td>Similarly, $x^{-2} = \frac{1}{x^2}$</td>
</tr>
<tr>
<td>$x^m \cdot x^n = x^{m+n}$</td>
<td>$x^3 \cdot x^7 = x^{10}$</td>
<td>$x^m \cdot x^n = (x \cdots x) \cdot (x \cdots x) = \underbrace{x \cdots x}<em>{m \text{ times}} \cdot \underbrace{x \cdots x}</em>{n \text{ times}} = x^{m+n}$ times</td>
</tr>
<tr>
<td>$x^m \div x^n = x^{m-n}$</td>
<td>$\frac{5^4}{5^2} = 5^2 = 5$</td>
<td>$x^m \div x^n = \underbrace{x \cdots x}<em>{m \text{ times}} \div \underbrace{x \cdots x}</em>{n \text{ times}} = x^{m-n}$</td>
</tr>
<tr>
<td>$(x^m)^n = x^{mn}$</td>
<td>$(2^3)^2 = 2^6$</td>
<td>$(2^a)^b = 2^a \cdot 2^a \cdot 2^a = 2^{a+b+a} = 2^3$</td>
</tr>
<tr>
<td>$(xy)^n = x^n y^n$</td>
<td>$(xy)^3 = x^3 y^3$</td>
<td>$(xy)^3 = (xy)(xy)(xy) = (x \cdot x \cdot x)(y \cdot y \cdot y) = x^3 y^3$ rearrange the order</td>
</tr>
<tr>
<td>$(\frac{x}{y})^n = \frac{x^n}{y^n}$</td>
<td>$(\frac{x}{y})^3 = \frac{x^3}{y^3}$</td>
<td>$(\frac{x}{y})^3 = (x \cdot y^{-1})^3 = x^3 \cdot y^{-3} = x^3 \cdot \frac{1}{y^3}$</td>
</tr>
<tr>
<td>$x^{\frac{1}{n}} = \sqrt[n]{x}$</td>
<td>$x^{\frac{1}{3}} = \sqrt[3]{x}$</td>
<td>$x^{\frac{1}{n}} = x = x = x$</td>
</tr>
<tr>
<td>$\frac{x^{\frac{1}{n}}}{y^{\frac{1}{n}}} = \sqrt[n]{\frac{x}{y}}$</td>
<td>$\frac{\sqrt[3]{3}}{\sqrt[3]{2}} = \frac{\sqrt[3]{9}}{\sqrt[3]{8}}$</td>
<td>So $x^{\frac{1}{n}}$ raised to $n^{th}$ power is $x$, which means that $x^{\frac{1}{n}} = \sqrt[n]{x}$.</td>
</tr>
<tr>
<td>$x^{\frac{m}{n}} = \sqrt[n]{x^m}$</td>
<td>$\sqrt[3]{5^2} = \sqrt[3]{25} = 5^{\frac{2}{3}}$</td>
<td>$x^{\frac{m}{n}} = (x^m)^{\frac{1}{n}} = \sqrt[n]{x^m}$.</td>
</tr>
<tr>
<td>$x^{-n} = \frac{1}{x^n}$</td>
<td>$\frac{1}{5^3} = \frac{1}{5^3}$</td>
<td>$x^{-n} = (x^{-1})^n = (\frac{1}{x})^n = \frac{1}{x^n}$.</td>
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</tbody>
</table>

**Example:** Simplify $\left(\sqrt[4]{5} \cdot 5^2\right)^3$.

**Solution:**

\[
\left(\sqrt[4]{5} \cdot 5^2\right)^3 = \left(\frac{5^{\frac{1}{4}} \cdot 5^2}{5^4}\right)^3 = \left(\frac{5^{\frac{9}{4}}}{5^4}\right)^3 = \frac{5^{\frac{3}{4}}}{5^4} = 5^{\frac{-11}{4}} = \frac{5^{\frac{3}{2}}}{5^4} = 5^{\frac{3}{2}+6-4} = 5^{\frac{11}{2}}.
\]
The exponential function

\[ f(x) = e^x \] This is the most commonly used exponential function in calculus.

This "e" is a very special constant. It is just a particular number \( 2 < e < 3 \). The precise value of \( e \) is \( e = 2.71828... \) (goes on forever).

What makes \( f(x) = e^x \) more special compared to \( g(x) = 2^x, h(x) = 3^x, r(x) = 7^x \), etc?

All exponential functions of the form \( f(x) = a^x \) (with \( a > 1 \)) pass through the point \((0,1)\), since \( a^0 = 1 \), and they look roughly like.

So, we can consider the "tangent line" to this graph at the point \((0,1)\).

The graph of \( f(x) = e^x \) (so, \( a = e \)) is special because it is the only one where the slope of the tangent line at \((0,1)\) is \( 1 \). Compare this with \( f(x) = 2^x, f(x) = 3^x \).