Let's recall the Fundamental Theorem of Calculus (FTC): Suppose that \( f(x) \) is a function with anti-derivative \( F(x) \) (recall this means that \( F'(x) = f(x) \)). Then:

\[
\int_{\alpha}^{\beta} f(x) \, dx = F(\beta) - F(\alpha)
\]

Recall that \( \int_{\alpha}^{\beta} f(x) \, dx \) is the **signed area** under the graph of \( f(x) \). Some examples:

\[
\int_{-1}^{3} f(x) \, dx = 5 \text{ means that the shaded region on the left has area 5.}
\]

\[
\int_{-1}^{3} g(x) \, dx = -5 \text{ indicates that the graph of } g(x) \text{ is underneath the x-axis. The shaded region has area } \left| \int_{-1}^{3} g(x) \, dx \right| = |-5| = 5.
\]

**Example of FTC:**

\[
\int_{1}^{3} x^2 \, dx = \frac{x^3}{3} \bigg|_{1}^{3} = \frac{3^3}{3} - \frac{1^3}{3} = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}
\]

This area is exactly \( \frac{26}{3} \)

But why does this method work? It seems magical that the area under a curve is related to derivatives!
In this lecture, we will prove FTC.
Given a reasonably nice function \( f(x) \)
(for our purposes, we assume \( f \) is continuous),
we will construct an explicit anti-derivative function \( F(x) \). We define

\[
F(x) = \int_{0}^{x} f(t) \, dt = \text{Area under } f(t) \\
\text{from } t=0 \text{ to } t=x.
\]

Let's understand why \( F(x) \) is actually a function.
For each input \( x \), \( F(x) \) outputs the (signed) area under the graph of \( f(t) \) between \( t=0 \) and \( t=x \). For example,

\[
F(3) = \text{shaded area in graph}
\]

\[
F(5) = \text{shaded area in graph}
\]

By the way, in definition of \( F(x) \),

\[
F(x) = \int_{0}^{x} f(t) \, dt,
\]

there is nothing special about this zero. We could have chosen any fixed constant.
Now that we understand why \( F(x) = \int_0^x f(t) \, dt \) is actually a function, we will prove:

**Claim**: \( F'(x) = f(x) \)

The significance of this claim is that we have constructed an anti-derivative for \( f(x) \). Indeed, the assertion \( F'(x) = f(x) \) literally means that an anti-derivative of \( f(x) \) is \( F(x) \).

**Proof of the Claim**: We will use the limit definition of derivative.

\[
F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}
\]

Observe that:

\( F(x+h) = \) shaded area in

In symbols:

\[
F(x+h) = \int_0^{x+h} f(t) \, dt
\]

\( F(x) = \) shaded area in

In symbols:

\[
F(x) = \int_0^x f(t) \, dt
\]
As a result:

\[ F(x+h) - F(x) = \text{shaded area in} \]

In symbols,

\[ F(x+h) - F(x) = \int_{x}^{x+h} f(t) \, dt \]

When \( h \) is small, the area above can be approximated by a rectangle of height \( f(x) \) and width \( (x+h) - x = h \). Indeed, let's zoom:

As \( h \to 0 \), this approximation in fact becomes perfect. The area of the rectangle is \( \frac{f(x) \cdot h}{\text{height}} \cdot \text{width} \)

\[ F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{f(x) \cdot h}{h} = f(x) \]

This shows \( F'(x) = f(x) \), and finishes the proof of the claim.
Now we are ready to prove FTC:

\[ F(b) - F(a) = \int_0^b f(t)dt - \int_0^a f(t)dt \]

\[ = \int_a^b f(t)dt \]

because \[ \int_0^b f(t)dt \]

So the difference is \[ \int_a^b f(t)dt - \int_0^a f(t)dt = \int_a^b f(t)dt \]

We have shown that:

\[ \int_a^b f(t)dt = F(b) - F(a) \]

Since "t" is a dummy variable, we can replace it with any other variable, say x. So:

\[ \int_a^b f(x)dx = F(b) - F(a) \]

which is exactly the statement of FTC.

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