Last time: We approximated the area under the curve $f(x) = x^2$ between $x=1$ and $x=3$ by constructing rectangles underneath.

**Attempt 1** (left-endpoints, 2 rectangles)

$$f(x) = x^2$$

$$R_1 + R_2 = 1 \cdot [1^2 + 2^2] = 5$$

$\uparrow$  \hspace{2cm} $\swarrow$  \hspace{2cm} $\swarrow$

width \hspace{2cm} heights

**Attempt 2** (right-endpoints, 2 rectangles)

$$R_1 + R_2 = 1 \cdot [2^2 + 3^2] = 13$$

$\uparrow$  \hspace{2cm} $\swarrow$  \hspace{2cm} $\swarrow$

width \hspace{2cm} heights

Combining Attempts 1 & 2, we see that the true area $A$ under the curve $f(x) = x^2$ between $x=1$ and $x=3$ satisfies $5 < A < 13$.

**Attempt 3**

Left endpoints, 4 rectangles.

$$R_1 + R_2 + R_3 + R_4 = \frac{1}{2}(1^2 + 1.5^2 + 2^2 + 2.5^2) = 6.75$$

**Attempt 4**

Right endpoints, 4 rectangles.

$$R_1 + R_2 + R_3 + R_4 = \frac{1}{2}(1.5^2 + 2^2 + 2.5^2 + 3^2) = 10.75$$

Combining Attempts 3 & 4, we see that $6.75 < A < 10.75$

**Moral:** More rectangles $\rightarrow$ Better approximation!
We will formalize this procedure:

We have the given interval \([a, b] = [1, 3]\) and we are considering the area underneath \(f(x) = x^2\) on this interval \([1, 3]\).

Suppose we want to approximate the area using \(n\) rectangles.

\[
\frac{1}{n} \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right) \Delta x
\]

Then the width of each rectangle should be \(\Delta x = \frac{b-a}{n} = \frac{3-1}{n}\).

What about the heights? Let's use left endpoints!

- Height of the first rectangle = \(1^2\)
- Height of the second rectangle = \((1 + \frac{2}{n})^2\)
- Height of the third rectangle = \((1 + \frac{4}{n})^2\)
  
  ... 

Height of the \(n^{th}\) rectangle = \((1 + \frac{2(n-1)}{n})^2\)

Thus, the total sum of the areas is

\[
S_n = \frac{1}{n} \sum_{i=1}^{n} \left( 1^2 + (1 + \frac{2}{n})^2 + (1 + \frac{4}{n})^2 + \ldots + (1 + \frac{2(n-1)}{n})^2 \right)
\]

\(S_n\) is called a \textbf{left Riemann sum}.

In the previous page, we already computed that \(S_2 = 5\) and \(S_4 = 6.75\). As \(n\) increases, \(S_n\) gets closer and closer to the true area.
In fact, as \( n \to \infty \), the rectangles become thinner and thinner (they have width \( \frac{1}{n} \)), and in the limit the rectangles perfectly approximate the area. Thus,

\[
\text{The area under the graph of } f(x) = x^2 = \lim_{n \to \infty} S_n. \\
\text{between } x=1 \text{ and } x=3
\]

More generally, if we want to find the area under a general function \( f(x) \) on \([a, b]\), we can perform the following:

**Graph of \( f(x) \)**

- **Left-endpoints**
  - Width of each rectangle = \( \Delta x = \frac{b-a}{n} \).
  - Sum of the areas = \( \sum_{i=0}^{n-1} f(x_i) \Delta x \)

- **Right endpoints**
  - Sum of the areas = \( \sum_{i=1}^{n} f(x_i) \Delta x \)

\[= \left\{ \begin{array}{ll}
\text{Left Riemann Sum} \\
\text{Right Riemann Sum}
\end{array} \right. \]
As \( n \to \infty \), the left Riemann sum and right Riemann sum converge to the same number, namely towards the true area under the curve \( f(x) \) for \( a \leq x \leq b \):

\[
\text{Area under } f(x) \text{ for } a \leq x \leq b = \lim_{{n \to \infty}} \sum_{{i=0}}^{n-1} f(x_i) \Delta x
\]

This has another name: integral!

And it is written as \( \int_a^b f(x) \, dx \).

You can think of the rectangles becoming thinner and thinner as \( n \to \infty \), and in the limit they become infinitesimally thin, so \( \Delta x \to dx \) and we are considering heights of rectangles of width 0 (crazy, right?), and that is why the height is \( f(x) \) for each point \( x \).

The point of the above definition is that:

\[
\int_a^b f(x) \, dx = \lim_{{n \to \infty}} \sum_{{i=0}}^{n-1} f(x_i) \Delta x
\]