Here is a scary derivative problem:

Find the derivative of \( f(x) = x^x \).

Wrong answers:

\[(x^x)' = x \cdot x^{-1} \quad \text{FALSE} \]

Cannot apply power rule \((x^n)' = nx^{n-1}\) here. \([n \text{ is fixed constant!}]\)

\[(x^x)' = x \cdot \ln x \quad \text{FALSE} \]

Cannot apply the rule \((b^x)' = b^x \cdot \ln(b)\) here. \([b \text{ is a fixed constant!}]\)

The tricky thing about \(x^x\) is that both the base ("x") and the exponent ("x") are variables!

So, how do we find the derivative of \(x^x\)?

Idea: Use the rule \(e^{\ln(x)} = x\) to get

\[e^{\ln(x^x)} = x^x \quad \text{just substitute } \ln(x) = x^x\).

Therefore, instead of finding derivative of \(x^x\), we can focus on finding derivative of \(e^{\ln(x^x)}\), which will be doable. Note that

\[e^{\ln(x^x)} = e^{x \ln x} \quad \text{using the rule } \ln(x^x) = x \ln x.\]

And we can find the derivative of \(e^{x \ln x}\) using the chain rule: \( (e^{x \ln x})' = e^{x \ln x} \cdot (x \ln x)' \).

We need to use product rule for \((x \ln x)'\).
We have \((x\ln x)' = x' \ln x + x \cdot (\ln x)'
\[
= 1 \cdot \ln x + x \cdot \left(\frac{1}{x}\right)
\]
\[
= \ln x + 1 = 1 + \ln x.
\]

Thus, \((e^{x \ln x})' = e^{x \ln x} \cdot (x \ln x)'
\[
= e^{x \ln x} \cdot (1 + \ln x)
\]
\[
= e^{\ln(x^x)} \cdot (1 + \ln x)
\]
\[
= x^x (1 + \ln x).
\]

We conclude that
\[
(x^x)' = (e^{\ln(x^x)})' = (e^{x \ln x})' = x^x (1 + \ln x).
\]

So \(\boxed{(x^x)' = x^x (1 + \ln x)}\)

Extra practice using this method:
1) Find the derivative of \(x \sin x\).
2) Find the derivative of \((\sin x)^{\sin x}\).
3) Find the derivative of \(x^{x^2}\).

Hint: For 1), write \(x \sin x = e^{\ln(x \sin x)} = e^{\ln(x) + \ln(\sin x)}\) and now use the chain rule to find the derivative of \(e^{\ln(x) + \ln(\sin x)}\). The same idea applies to 2) and 3) as well.
Part I: Differential Calculus

Part II: Integral Calculus.

Example 1: Consider the graph of $f(x) = x^2$ (parabola)

Our goal is to find the area of the shaded region above. In other words, we would like to find the area under the curve $f(x) = x^2$ between $x=1$ and $x=3$.

Idea: We can approximate the area by rectangles! Finding area of a rectangle is easy (base x height). So, we will "cover" the area by rectangles and obtain an approximate area.

Let's use two rectangles for now. There are several ways of placing the rectangles. We will discuss 2 such methods: left-end points, and right end-points. See next page.
Attempt 1: Placing the left corners of rectangles on the graph.

Area of the two rectangles:
\[ \frac{1 \cdot 1}{\text{width}} + \frac{1 \cdot 4}{\text{width}} = 5 \]
\[ \frac{\text{height}}{\text{height}} \]
This is an underestimate for the actual area under the curve.
(some space not covered by the two rectangles!)

Attempt 2: Placing the right corners of rectangles on the graph.

Area of the 2 rectangles:
\[ \frac{1 \cdot 4}{\text{width}} + \frac{1 \cdot 9}{\text{width}} = 13 \]
This is an overestimate for the actual area under the curve.
(because some extra space is covered by the 2 rectangles).

So the exact answer for the area is somewhere between 5 and 13. How can we find better approximations? Idea: Use more rectangles.
We can use 4 rectangles. We will draw two pictures again, one for left-end points and one for right end points.

**Attempt 1**: Left-endpoints.

\[ f(x) = x^2 \]

![Diagram of left-endpoints approximation]

Total sum = \( R_1 + R_2 + R_3 + R_4 = 0.5 \left( \frac{1^2 + 1.5^2 + 2^2 + 2.5^2}{\text{width}} \right) = 6.75 \)

This is again an underestimate for the true area.

**Attempt 2**: Right endpoints.

Total sum = \( R_1 + R_2 + R_3 + R_4 = 0.5 \left( \frac{1.5^2 + 2^2 + 2.5^2 + 3^2}{\text{width}} \right) = 10.75 \)

This is again an overestimate for the true area.

Now, we know that the area under the curve \( f(x) = x^2 \) (from \( x=1 \) to \( x=3 \)) is between 6.75 and 10.75.