Maximizing / minimizing functions

Suppose that \(f(x)\) is a function defined on a closed interval \([a, b]\). The graph of \(f(x)\) could look like:

We want to find the points \(x\) for which the function value \(f(x)\) is maximized/minimized.

Local max / global max / local min / global min

We say that the function \(f\) has a local max at \(x = c\) if \(f(c) \geq f(x)\) for all \(x\) in a (small) interval around \(c\). (Similar definition holds for local min.)

Question: Where can we find local minimums and local maximums of a given function?

Answer: Look for points where \(f'(x) = 0\).
The points where the local max/mins occur have zero slope for their tangent line (horizontal tangent line), which means the derivative at these points is zero.

Steps for finding global max/min of a given function $f(x)$ on the interval $[a,b]$.

1. **Step 1**: Find $f'(x)$, and solve the equation $f'(x)=0$. In other words, find all values of $x$ such that $f'(x)=0$. These values are called "critical values" of $f$.

2. **Step 2**: Plug in the critical values of $f(x)$ into $f$ and find the corresponding $y$-values.

3. **Step 3**: Test the function at the endpoints.

4. **Step 4**: Take the largest/smallest value.

**Example**: $f(x) = x^2 - 2x - 3$ on $[-5,5]$.

1. **Step 1**: $f'(x) = 2x - 2$. Set $f'(x) = 0 \rightarrow 2x - 2 = 0$ $\rightarrow 2x = 2 \rightarrow x = 1$

2. **Step 2**: $f(1) = 1^2 - 2(1) - 3 = -4$.

3. **Step 3**: Endpoints: $f(-5) = (-5)^2 - 2(-5) - 3 = 32$, $f(5) = 5^2 - 2(5) - 3 = 12$.

4. **Step 4**: $f(1) = -4 \leftarrow$ global min, $f(-5) = 32 \leftarrow$ global max.
Problem: A farmer wants to fence an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this in order to minimize the cost of the fence?

Solution: Draw a picture:

\[
\begin{array}{c}
\hspace{1cm}
\end{array}
\]

Constraint: \( x \cdot y = 1.5 \) (million square feet)

We are trying to minimize the fence:

Fence length: \( 2x + 3y \). \( \leftarrow \) we want to minimize this quantity.

Substitute \( y = \frac{1.5}{x} \) into to get:

\[
f(x) = 2x + 3 \left( \frac{1.5}{x} \right) = 2x + \frac{4.5}{x} = 2x + (4.5)x^{-1}
\]

This is the function we want to minimize.

\[
f'(x) = 2 + (4.5)(-x^{-2}) \implies f'(x) = 0
\]

\[
2 = (4.5)x^{-2} = \frac{4.5}{x^2}
\]

\[
\iff x^2 = \frac{4.5}{2} = 2.25 \rightarrow x = 1.5
\]

\[
\rightarrow y = \frac{1.5}{x} = \frac{1.5}{1.5} = 1.
\]