Instructions. Please work on the following problems in groups, while the TAs circulate. While you will not need to submit your answers, make sure to show your progress to the TAs on every problem.

(1) Look over the annotated midterm solutions. Go over each problem carefully in groups, and see if you understand the solution given there. Please address each comment written in a speech balloon. If you incorrectly solved the problem, do you feel ready to solve the problem now? If you have solved the problem correctly, you can explain your reasoning to your peers. You can also see if you could have improved your solution. Maybe you forgot to write \( \lim_{h \to 0} \) at each step of the solution, etc.

(2) Find the derivative of the following function:
\[
f(x) = (\sin(e^x + x))^8
\]

(3) Find the derivative of the following function:
\[
g(x) = \ln(\cos(x^3 + 3x^2 + 7x + 1))
\]

(4) Find the derivative of the following function:
\[
h(x) = \sqrt{(e^x + 5x^2)^3 + 7\sin x}
\]

(5) Our goal is to approximate the area under the curve \( f(x) = x^2 \) between \( x = 1 \) and \( x = 5 \). Divide the interval \([1, 5]\) into 8 equal pieces and construct rectangles of width \( 4/8 = 1/2 \) with left endpoints on the graph, and approximate the area under the curve by summing the areas of these 8 rectangles. Is this an underestimate or an overestimate for the true area? Draw a picture!

Next, do the same problem again but this time draw your rectangles so that their right endpoints touch the graph of \( f(x) = x^2 \). Approximate the area under the curve by summing the areas of these 8 rectangles. Is this an underestimate or an overestimate for the true area? Draw a picture!
a) State the domain and range of \( f(x) = 2x - 1 \).

b) State the domain and range of \( f(x) = |x| \) (the absolute value function).

c) Find the domain of the following function: \( f(x) = \frac{x^2}{x^2 - 4x - 21} \).

Solution.

1. The domain of \( f(x) = 2x - 1 \) is all real numbers. The range of \( f(x) = 2x - 1 \) is all real numbers. Instead of writing “all real numbers”, you can also write \((-\infty, \infty)\).

2. The domain of \( f(x) = |x| \) is all real numbers. The range of \( f(x) = |x| \) is \([0, \infty)\).

3. The domain of \( f(x) = \frac{x^2}{x^2 - 4x - 21} \) consists of all real numbers \( x \) such that the denominator \( x^2 - 4x - 21 \) is not zero. So we need to figure out when the denominator is 0, which means we need to figure out when \( x^2 - 4x - 21 = 0 \).

Since \( x^2 - 4x - 21 = (x - 7)(x + 3) \), we see that \( (x + 3)(x - 7) = 0 \) exactly when \( x = -3 \) or \( x = 7 \). In other words, when \( x = -3 \) or \( x = 7 \), the denominator is zero, and we cannot evaluate \( f(x) \) at these two \( x \)-values.

Thus, the domain of \( f(x) \) is all real numbers except for \( x = -3 \) and \( x = 7 \). We can also write the domain using the interval notation:

\[
\text{Domain of } f = (-\infty, -3) \cup (-3, 7) \cup (7, \infty)
\]

Can you explain this geometrically by looking at the graph of \( f(x) = 2x - 1 \)? Why is the range all real numbers?

Can you explain this geometrically by looking at the graph of \( f(x) = |x| \)? Why are negative numbers not in the range?

Does this notation make sense?

Is factoring step okay? If you don’t know factoring, are you familiar with the quadratic formula?
Evaluate the following limits. If the limit doesn’t exist, then explain why.

a) \( \lim_{x \to 4} \frac{x - 4}{\sqrt{x + 5} - 3} \)

b) \( \lim_{x \to 3} f(x) \) where \( f(x) = \begin{cases} 
    x^2 + 1 & \text{for } x < 3 \\
    9 & \text{for } x = 3 \\
    3x & \text{for } x > 3
\end{cases} \)

Solution.

a) If we substitute \( x = 4 \), we will get \( 0/0 \) which means we have to do more work in order to figure out the limit. The trick is to multiply the fraction by the conjugate of \( \sqrt{x + 5} - 3 \), which is \( \sqrt{x + 5} + 3 \). We have:

\[
\lim_{x \to 4} \frac{x - 4}{\sqrt{x + 5} - 3} = \lim_{x \to 4} \frac{x - 4}{\sqrt{x + 5} - 3} \cdot \frac{\sqrt{x + 5} + 3}{\sqrt{x + 5} + 3} \\
= \lim_{x \to 4} \frac{(x - 4)(\sqrt{x + 5} + 3)}{(\sqrt{x + 5} - 3)(\sqrt{x + 5} + 3)} \\
= \lim_{x \to 4} \frac{(x - 4)(\sqrt{x + 5} + 3)}{(x + 5) - 9} \\
= \lim_{x \to 4} \frac{(x - 4)(\sqrt{x + 5} + 3)}{x - 4} \\
= \lim_{x \to 4} \frac{(x + 5) + 3}{1} \\
= \sqrt{4 + 5} + 3 = \sqrt{9} + 3 = 3 + 3 = 6
\]

b) We compute the left-side limit and the right-side limit of the function:

\[
\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 + 1) = 3^2 + 1 = 10 \\
\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 3x = 3 \cdot 3 = 9
\]

Since \( \lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x) \), the overall limit \( \lim_{x \to 3} f(x) \) does not exist.

In the solution, we didn’t even care that \( f(3) = 9 \). Why? Would our answer change if we knew that \( f(3) = 15 \)?
Let \( f(x) = x^2 - 2 \).

a) Use the limit definition of the derivative to find \( f'(x) \).

b) Find the slope of the tangent line to \( f(x) \) at \( x = 1 \).

c) Find the equation of the tangent line to \( f(x) \) at \( x = 1 \).

a) Using the limit definition of derivative, we get

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - 2 - (x^2 - 2)}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 2 - x^2 + 2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} \frac{h(2x + h)}{h} = \lim_{h \to 0} 2x + h = 2x + 0 = 2x\]

Thus, \( f'(x) = 2x \).

If you made a mistake in the midterm, where was your error? Did you forget the limit definition of derivative? Did you have trouble with the algebra? Can you solve the same problem for \( f(x) = 1/x \)?

b) The slope of the tangent line to \( f(x) \) at \( x = 1 \) is equal to \( f'(1) \). Using part a) above, we get \( f'(1) = 2 \cdot 1 = 2 \). So the slope of the tangent line is 2.

Does this make sense? If you were asked to find the slope of the tangent line at \( x = 5 \), what would you do?

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b) The slope of the tangent line to \( f(x) \) at \( x = 1 \) is equal to \( f'(1) \). Using part a) above, we get \( f'(1) = 2 \cdot 1 = 2 \). So the slope of the tangent line is 2.

Does this make sense? If you were asked to find the slope of the tangent line at \( x = 5 \), what would you do?

c) The equation of the tangent line is \( y = mx + b \) where \( m \) is the slope. From part b), we know that \( m = 2 \). So \( y = 2x + b \). In order to find \( b \), we need a point on the tangent line. We can just take the tangency point, which is \((1, f(1))\). Since \( f(1) = 1^2 - 2 = -1 \), the point on the line is \((x, y) = (1, -1)\). Substituting the coordinates of this point into \( y = 2x + b \), we obtain \(-1 = 2 \cdot 1 + b \) so \(-1 = 2 + b \) which allows us to solve \( b = -3 \).

Thus, the equation of the tangent line at \( x = 1 \) is given by \( y = 2x - 3 \).

Can you see why we can plug in the point \((1, f(1))\) in order to find the value of \( b \)? Can you find the equation of the line using the point-slope formula? Sketch the graph of \( f(x) \) for this problem, and draw the tangent line at \( x = 1 \).
Use the rules of differentiation to find the derivatives of the following functions.

a) \( f(x) = e^x \sin x + 3 \ln x \)

b) \( g(x) = \frac{2x + 1}{x - 1} \)

c) \( h(x) = \sin(x^2 + x + 1) + x^{3/2} + e^x \)

Solution.

a) We have

\[
  f'(x) = (e^x)' \cdot \sin x + e^x \cdot (\sin x)' + (3 \ln x)' = e^x \sin x + e^x \cos x + \frac{3}{x}
\]

What rule did we use here? Why is the derivative of 3 ln x is 3/x? Some people thought the derivative of 3 ln x is 1/(3x). Can you see why this is incorrect?

b) Using the Quotient Rule,

\[
  g'(x) = \frac{(2x + 1)' \cdot (x - 1) - (2x + 1) \cdot (x - 1)'}{(x - 1)^2} \\
  = \frac{2(x - 1) - (2x + 1) \cdot 1}{(x - 1)^2} \\
  = \frac{2x - 2 - 2x - 1}{(x - 1)^2} = \frac{-3}{(x - 1)^2}
\]

If you made a mistake on this problem, where was your error? Did you state the quotient rule incorrectly? Did you incorrectly find the derivatives of 2x+1 and x-1? Algebra mistakes while simplifying?

c) We have

\[
  h'(x) = \cos(x^2 + x + 1) \cdot (x^2 + x + 1)' + \frac{3}{2} x^{1/2} + 0 \\
  = \cos(x^2 + x + 1) \cdot (2x + 1) + \frac{3}{2} \sqrt{x}
\]

What rule did we use for the first term in h(x)? What about the second term? Where did the square root come from? What happened to the term e^{\pi}i? Some people ignored it completely, but to receive full credit, you should have written +0 to indicate that you realize it is a constant. Remember always to show your work!