1 Formulas and conceptual concepts

Definitions (explain in words)

- What is the normal vector to a surface at a point?
- What is a directional derivative at a point?
- What is a tangent plane?

Formulas

- Gradient
- Differential
- Tangent plane
- Normal vector to a surface at a point
- Directional derivative
- Direction of a vector tangent to a surface
- Can you generalize all of the above for a three variable function \( w = f(x, y, z) \)?

Conceptual ideas (For some pictures see the following link)

- What is the meaning of the direction of the gradient?
- What is the meaning of the magnitude of the gradient?
- For a surface \( z = f(x, y) \) in \( \mathbb{R}^3 \) the gradient vector only has two coordinates. How can you visualize this gradient vector in \( \mathbb{R}^3 \)? How does this gradient vector compare to the level curve?
- Explain why for a surface \( c = F(x, y, z) \) (\( c \) is a constant), like \( 9 = x^2 + y^2 + z^2 \), the gradient \( \nabla F \) is the normal vector.
- Find one reason why your formula for a normal vector at a point is indeed normal to the surface.
- Explain why the directional derivative formula is true. (Hint: differentials)
- Given a direction \( \vec{v} \) (which may or may not be a unit vector) how does the formula for the directional derivative change?
- Explain why directional vector for the tangent line at a point makes sense. How does it relate to the directional derivative?
- Given a direction \( \vec{v} \) (which may or may not be a unit vector) how does the formula for the directional vector for the tangent line at a point change?
2 Practice problems

– The secondary text # 1 has solutions for nearly all problems. If you would like more similar practice you can try all the problems in the section.
– For problems with guided solutions look at the mentioned examples from the main text. Problems are restated here.

Straight forward calculations

• Secondary text # 1 Section 14.5 # 1, 3, 6, 10, 12, 15 (page 370 of text)

• Example 422 from main text: Let \( z = 14 - x^2 - y^2 \) and let \( P = (1, 2) \). Find the directional derivative of \( f \), at \( P \), in the following directions: Towards \( Q = (3, 4) \). In the direction \( \langle 2, -1 \rangle \). Towards the origin.

• Example 423 from main text: Let \( f(x, y) = \sin x \cos y \) and let \( P = (\pi/3, \pi/3) \). Find the directions of maximal/minimal increase, and find a direction where the instantaneous rate of \( z \) change is 0.

• Example 427 from main text: Find the lines tangent to the surface \( z = \sin x \cos y \) at \( (\pi/2, \pi/2) \) in the \( x \) and \( y \) directions and also in the direction of \( \vec{v} = \langle -1, 1 \rangle \).

• Example 429 from main text: Find the equation of the normal line to \( z = -x^2 - y^2 + 2 \) at \( (0, 1) \).

• Example 432 from main text: Find the equation tangent plane to \( z = -x^2 - y^2 + 2 \) at \( (0, 1) \).

Harder problems / Applications

• Secondary text # 1 Section 14.5 # 8, 16, 18 (page 370 of text)

• Example 425 from main text: Consider the surface given by \( f(x, y) = 20 - x^2 - 2y^2 \) . Water is poured on the surface at \( (1, 1/4) \). What path does it take as it flows downhill?

• Example 430 from main text: Let \( f(x, y) = 2 - x^2 - y^2 \) and let \( Q = (2, 2, 2) \). Find the distance from \( Q \) to the surface defined by \( f \).

• Example 431 from main text: Let \( f(x, y) = x - y^2 + 3 \). Let \( P = (2, 1, f(x, 2, 1)) = (2, 1, 4) \). Find points \( Q \) in space that are 4 units from the surface of \( f \) at \( P \). That is, find \( Q \) such that \( ||\vec{PQ}|| = 4 \) and \( \vec{PQ} \) is orthogonal to \( f \) at \( P \).

• Given the directional derivatives \( D_{\vec{u}}f(0, 0) = 0 \) and \( D_{\vec{v}}f(0, 0) = 1 \) for \( \vec{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle \) and \( \vec{v} = \langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle \) what is \( f_x(0, 0) \) and \( f_y(0, 0) \)?
Contour diagram examples
For the contour diagrams answer the following at the specified point.

(a) Estimate $f_x$ and $f_y$.

(b) Estimate the direction of $\nabla f$, $||\nabla f||$ and $\nabla f$.

(c) Estimate $D_{\bar{u}}f$ for $\bar{u} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.

(d) Estimate the direction of steepest slope and the directions of zero slope.
   
   1. At any point.

   2. At the point $P = (1, 2)$. The green level is when $z = 0$, the blue levels are the positive levels incrementing by one and the red levels are the negative levels.