1 Formulas and conceptual concepts

Definitions (explain in words)

- What is the normal vector to a surface at a point?
- What is a directional derivative at a point?
- What is a tangent plane?

Formulas

- Gradient
- Differential
- Tangent plane
- Normal vector to a surface at a point
- Directional derivative
- Direction of a vector tangent to a surface
- Can you generalize all of the above for a three variable function $w = f(x, y, z)$?

Conceptual ideas (For some pictures see the following link)

- What is the meaning of the direction of the gradient?
- What is the meaning of the magnitude of the gradient?
- For a surface $z = f(x, y)$ in $\mathbb{R}^3$ the gradient vector only has two coordinates. How can you visualize this gradient vector in $\mathbb{R}^3$? How does this gradient vector compare to the level curve?
- Explain why for a surface $c = F(x, y, z)$ ($c$ is a constant), like $9 = x^2 + y^2 + z^2$, the gradient $\nabla F$ is the normal vector.
- Find one reason why your formula for a normal vector at a point is indeed normal to the surface.
- Explain why the directional derivative formula is true. (Hint: differentials)
- Given a direction $\vec{v}$ (which may or may not be a unit vector) how does the formula for the directional derivative change?
- Explain why directional vector for the tangent line at a point makes sense. How does it relate to the directional derivative?
- Given a direction $\vec{v}$ (which may or may not be a unit vector) how does the formula for the directional vector for the tangent line at a point change?
2 Practice problems

– The secondary text # 1 has solutions for nearly all problems. If you would like more similar practice you can try all the problems in the section.
– For problems with guided solutions look at the mentioned examples from the main text. Problems are restated here.

Straight forward calculations

• Secondary text # 1 Section 14.5 # 1, 3, 6, 10, 12, 15 (page 370 of text)

• Example 422 from main text: Let $z = 14 - x^2 - y^2$ and let $P = (1,2)$. Find the directional derivative of $f$, at $P$, in the following directions: Towards $Q = (3, 4)$. In the direction $\langle 2, -1 \rangle$. Towards the origin.

• Example 423 from main text: Let $f(x,y) = \sin x \cos y$ and let $P = (\pi/3, \pi/3)$. Find the directions of maximal/minimal increase, and find a direction where the instantaneous rate of $z$ change is 0.

• Example 427 from main text: Find the lines tangent to the surface $z = \sin x \cos y$ at $(\frac{\pi}{2}, \frac{\pi}{2})$ in the $x$ and $y$ directions and also in the direction of $\vec{v} = \langle -1, 1 \rangle$.

• Example 429 from main text: Find the equation of the normal line to $z = -x^2 - y^2 + 2$ at $(0,1)$.

• Example 432 from main text: Find the equation tangent plane to $z = -x^2 - y^2 + 2$ at $(0,1)$.

Harder problems / Applications

• Secondary text # 1 Section 14.5 # 8, 16, 18 (page 370 of text)

• Example 425 from main text: Consider the surface given by $f(x,y) = 20 - x^2 - 2y^2$. Water is poured on the surface at $(1,1/4)$. What path does it take as it flows downhill?

• Example 430 from main text: Let $f(x,y) = 2 - x^2 - y^2$ and let $Q = (2,2,2)$. Find the distance from $Q$ to the surface defined by $f$.

• Example 431 from main text: Let $f(x,y) = x - y^2 + 3$. Let $P = (2,1, f(x,2,1)) = (2,1,4)$. Find points $Q$ in space that are 4 units from the surface of $f$ at $P$. That is, find $Q$ such that $||PQ|| = 4$ and $PQ$ is orthogonal to $f$ at $P$.

• Given the directional derivatives $D_\vec{u}f(0,0) = 0$ and $D_\vec{v}f(0,0) = 1$ for $\vec{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ and $\vec{v} = \langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$ what is $f_x(0,0)$ and $f_y(0,0)$?
Contour diagram examples
For the contour diagrams answer the following at the specified point.

(a) Estimate $f_x$ and $f_y$.

(b) Estimate the direction of $\nabla f$, $||\nabla f||$ and $\nabla f$.

(c) Estimate $D_{\bar{u}}f$ for $\bar{u} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

(d) Estimate the direction of steepest slope and the directions of zero slope.

1. At any point.

2. At the point $P = (1, 2)$. The green level is when $z = 0$, the blue levels are the positive levels incrementing by one and the red levels are the negative levels.
Answers to non-book questions

- \( f_x(0, 0) = -\frac{3}{\sqrt{5}} \) and \( f_y(0, 0) = \frac{3}{\sqrt{5}} \)

1. (a) \( f_x = \frac{\Delta z}{\Delta x} = -\frac{1}{2} \) and \( f_y = \frac{\Delta z}{\Delta y} = \frac{3}{4} \).

(b) \( \nabla f = \langle -\frac{1}{2}, \frac{3}{4} \rangle \), direction as a unit vector is \( \frac{4}{\sqrt{13}} \langle -\frac{1}{2}, \frac{3}{4} \rangle \) and the magnitude is \( ||\nabla f|| = \frac{\sqrt{13}}{4} \).

(c) \( D_uf = \frac{5\sqrt{2}}{8} \).

(d) Direction of steepest slope is \( \nabla f = \langle -\frac{1}{2}, \frac{3}{4} \rangle \) and a direction of zero slope is \( \langle 3, 2 \rangle \). Note that the dot product is zero.

2. At the point \((1, 2)\)

(a) \( f_x = \frac{\Delta z}{\Delta x} = -2 \) and \( f_y = \frac{\Delta z}{\Delta y} = \frac{4}{3} \).

(b) \( \nabla f = \langle -2, \frac{4}{3} \rangle \), direction as a unit vector is \( \frac{3}{2\sqrt{13}} \langle -2, \frac{4}{3} \rangle \) and the magnitude is \( ||\nabla f|| = \frac{2\sqrt{13}}{3} \).

(c) \( D_uf = \frac{5\sqrt{2}}{3} \).

(d) Direction of steepest slope is \( \nabla f = \langle -2, \frac{4}{3} \rangle \) and a direction of zero slope is \( \langle 1, 2 \rangle \). Note that the dot product is close to zero, but isn’t zero because our gradient was an estimation.