The University of British Columbia
Midterm 2 MATH 310-201, Mar 18th 2016
Mathematics 310-201
Version 2

Closed book examination.

Last Name ___________ First ___________ Signature ___________

Student Number

Special Instructions:
No books, notes, or calculators are allowed.

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Student Conduct during Examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCCard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (a) speaking or communicating with other candidates, unless otherwise authorized;
  - (b) purposely exposing written papers to the view of other candidates or imaging devices;
  - (c) purposely viewing the written papers of other candidates;
  - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) - (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).
1. Label the following statements as TRUE or FALSE? You do not need to justify your answer.

1. If \( f, g : V \to V \) are orthogonal, so is \( f + g \).

   \[ \text{FALSE} \quad \text{orthogonal } \Rightarrow \text{length of columns of matrix is 1}. \]
   \[ \text{not preserved under addition}. \]

2. Every matrix over the complex numbers is diagonalizable.

   \[ \text{FALSE}. \quad \text{We had } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ as an Exercise}. \]

3. The Range and the Nullspace of a symmetric map are always orthogonal.

   \[ \text{TRUE}: f : V \to V, \quad \forall \mathbf{v} \in N(f) \Rightarrow <f(w), \mathbf{v}> \leq 0, \quad \forall \mathbf{w} \in R(f) \]

4. There is a symmetric matrix whose characteristic polynomial is \( p(t) = t^2 + 1 \).

   \[ \text{FALSE}. \quad \text{It has to have roots over the real numbers}. \]
2. Find all Eigenvalues and Eigenvectors of the $\mathbb{R}$-linear map $\mathbb{R}^3 \to \mathbb{R}^3$ whose matrix is given by

$$
\begin{pmatrix}
1 & 2 & 3 \\
0 & 4 & 5 \\
0 & 0 & 6
\end{pmatrix}.
$$

char poly: $\text{det} \left( \begin{pmatrix} 1-t & 2 & 3 \\
0 & 4-t & 5 \\
0 & 0 & 6-t \end{pmatrix} \right) = (1-t)(4-t)(6-t)$

$\Rightarrow$ Eigenvalues $1, 4, 6$

2 = 1:

$$
\begin{pmatrix}
0 & 2 & 3 \\
0 & 3 & 5 \\
0 & 0 & 5
\end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
$$

$\Rightarrow$ Nullspace $= \{ \lambda \cdot \begin{pmatrix} 0 \\
0 \\
0 \end{pmatrix} | \lambda \in \mathbb{R} \}$

2 = 4:

$$
\begin{pmatrix}
-3 & 2 & 3 \\
0 & 0 & 5 \\
0 & 0 & 3
\end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix}
1 & -\frac{2}{3} & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
$$

$\Rightarrow$ Eigenvector: $\begin{pmatrix} \frac{2}{3} \\
1 \\
0 \end{pmatrix}$

2 = 6:

$$
\begin{pmatrix}
-5 & 2 & 3 \\
0 & -2 & 5 \\
0 & 0 & 0
\end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix}
-5 & 0 & 8 \\
0 & -2 & 5 \\
0 & 0 & 0
\end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix}
1 & 0 & -\frac{5}{2} \\
0 & 1 & -\frac{5}{2} \\
0 & 0 & 0
\end{pmatrix}
$$

$\Rightarrow$ Eigenvector: $\begin{pmatrix} \frac{5}{2} \\
\frac{5}{2} \\
1 \end{pmatrix}$
3. Let \( s : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \) be the inner product given by \( s \left( \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix} \right) = 2aa' + ab' + ba' + 2bb' \). Find an orthonormal basis of \( \mathbb{R}^2 \) (equipped with this inner product).

A basis of \( \mathbb{R}^2 \) is given by \( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). Use Gram-Schmidt:

\[
e_1 = \frac{1}{\|e_1\|} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \|e_1\| = s(e_1, e_1)^{1/2} = \sqrt{2}.
\]

\[
e_1 = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

\[
\tilde{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - s(e_1, e_1) \cdot e_1
\]

\[
= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - s \left( \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - s \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}.
\]

\[
\|\tilde{e}_2\| = s \left( \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 1 \end{pmatrix} \right)^{1/2} = \left( \frac{1}{2} - 1 - 1 + 2 \right)^{1/2} = \left( \frac{1}{2} \right)^{1/2}.
\]

\[
\tilde{e}_2 = \sqrt{2} \cdot \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}.
\]

So we have found an ONB \( \tilde{e}_1, \tilde{e}_2 \).
4. Let $V$ be a finite-dimensional inner product space and let $f : V \to V$ be a linear map. Show that the Nullspace of $f^* \circ f$ and the Nullspace of $f$ are the same.

To show $N(f) = N(f^* \circ f)$

1. $v \in N(f) \Rightarrow f(v) = 0 \Rightarrow f^* f(v) = 0 \Rightarrow v \in N(f^* \circ f)$.

2. $v \in N(f^* \circ f) \Rightarrow f^* f(v) = 0 \Rightarrow \langle v, f^* f(v) \rangle = 0 \Rightarrow \langle f(v), f(v) \rangle = 0 \Rightarrow f(v) = 0 \Rightarrow v \in N(f)$. 

\text{pos.}
5. Let $V$ be a finite-dimensional inner product space. Let $e_1 \in V$ be given with $\|e_1\| = 1$. Define a linear map $f : V \to V$ by

$$w \mapsto w - 2\langle e_1, w \rangle \cdot e_1.$$ 

Show that $f$ is orthogonal.

Complete $e_1$ to an OUMB $e_1, \ldots, e_n$ of $V$. Let us compute the matrix of $f$ w.r.t. that basis:

$$f(e_1) = e_1 - 2\langle e_1, e_1 \rangle \cdot e_1 = -e_1,$$

$$f(e_j) = e_j - 2\langle e_1, e_j \rangle \cdot e_1 = e_j,$$

So the matrix is 

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = A.$$ 

And we have $A^* = A = A^{-1}$. So $f$ is orthogonal.
Please be aware that many exercises allow different non-unique solutions. For example, there are many ONB's that would work for (3).