The University of British Columbia
Final MATH 310-201, Apr 20th 2016
Mathematics 310-201
Version 2

Closed book examination.

Last Name Solutions First __________________ Signature __________________

Student Number

Special Instructions:
No books, notes, or calculators are allowed.

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Student Conduct during Examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  
  a. speaking or communicating with other candidates, unless otherwise authorized;
  b. purposely exposing written papers to the view of other candidates or imaging devices;
  c. purposely viewing the written papers of other candidates;
  d. using or having visible at the place of writing any books, papers, or other memory aid devices other than those authorized by the examiner(s); and,
  e. using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).
1. Label the following statements as TRUE or FALSE? You do not need to justify your answer.

1. A product of two elements in a field can only be equal to zero, if at least one of the factors is zero.

   True

2. Suppose the characteristic polynomial of \( f : V \to V \) is irreducible. Then 0, \( V \) are the only \( f \)-invariant subspaces of \( V \).

   True. If \( W \subset V \) is \( f \)-invariant, then we can look at the matrix of \( f: W \to W \), the char. poly. of that block is a factor of the char. poly. of \( f \).

   \[
   \begin{pmatrix}
   A & \ast \\
   0 & B
   \end{pmatrix}
   \]

3. Let \( W \subset V \) be a subspace and let \( V' \) be a vector space. Then any linear map \( W \to V' \) can be extended to a linear map \( V \to V' \).

   True

4. Let \( f : V \to V \) be a linear map and let \( V \) be finite dimensional. Let \( A \) be the matrix of \( f \) with respect to some basis. Then the sum of the diagonal entries of that matrix does not depend on the choice of the basis.

   True

5. Let \( V \) be an inner product space. Then any normal map \( V \to V \) is symmetric.

   False. (The other way around).
6. Let $f : V \to W, g : W \to V$ be two linear maps. If $f \circ g = id_W$, then $g \circ f = id_V$.

$$
\begin{pmatrix}
1 & 1 \\
0 & 0 \\
\end{pmatrix} \neq \begin{pmatrix} 1 \\
0 \\
\end{pmatrix} \quad \text{but} \quad \begin{pmatrix} 0 \\
0 \\
\end{pmatrix} \neq \begin{pmatrix}
1 & 1 \\
0 & 0 \\
\end{pmatrix}
$$

7. $1 + 1 \neq 0$ in any field.

\text{False}

8. The map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x + 1$ is linear.

\text{False}

9. If the Jordan normal form of a complex square matrix $A$ contains a block of size $\geq 2$, then $A$ cannot be diagonalizable.

\text{True} \quad \text{(Uniqueness of JNF)}

10. If $s : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is bilinear, then

$$
\mathbb{R}^2 \to \mathbb{R}, \quad \begin{pmatrix} a \\
b \\
\end{pmatrix} \mapsto s(a,b)
$$

is linear.

\text{False}
11. A basis is a linearly independent generating system.

True

12. Two matrices with the same characteristic polynomial must have the same determinant.

True

13. Let $V$ be a finite-dimensional vector space. A linear map $f : V \to V$ is invertible, if and only if its determinant is nonzero.

True

14. If $f : V \to W$ is linear, then $\dim(V) = \dim(N(f)) + \dim(R(f))$.

True

15. If $A^2 = 0$, then $A = 0$ (for a square matrix $A$).

False

16. Two matrices with the same characteristic polynomial are conjugate.

False (JNF)
17. The sum of two orthogonal maps is orthogonal

False

18. The product of two symmetric matrices is symmetric.

False
2. Let $f$ be the following map

$$f : \mathbb{R}^3 \to \mathbb{R}^2, \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a - c \\ b - 2c \end{pmatrix}$$

Find the matrix of $f$ with respect to the bases \( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) of $\mathbb{R}^3$ and \( \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) of $\mathbb{R}^2$.

\[
\begin{align*}
    f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
    f\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
    f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} = (+1) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-2) \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{align*}
\]
3. Find the inverse of the following matrix:

\[
\begin{pmatrix}
1 & 2 & 5 \\
2 & 1 & 4 \\
0 & 3 & 7
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 5 & 1 & 0 & 0 \\
0 & 3 & 7 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 5 & 1 & 0 & 0 \\
0 & 3 & 7 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 0 & 11 & -5 & -5 \\
0 & 1 & 0 & 14/3 & -7/3 & 2 \\
0 & 0 & 1 & -2 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 11 & -28/3 & -5 + 14/3 & -1 \\
0 & 1 & 0 & 14/3 & -7/3 & 2 \\
0 & 0 & 1 & -2 & 1 & 1
\end{pmatrix}
\]

So the inverse is:

\[
\begin{pmatrix}
5/3 & -1/3 & -1 \\
14/3 & -7/3 & 2 \\
-2 & 1 & 1
\end{pmatrix}
\]
4. Compute the determinant of the following matrix:

\[ A = \begin{pmatrix}
1 & 2 & 5 & 47 & 121 & 3 \\
2 & 1 & 4 & 1 & 1 & 1 \\
0 & 0 & 3 & 7 & 0 & 0 \\
0 & 0 & 2 & -4 & 0 & 0 \\
0 & 0 & 1 & 1 & 2 & 3 \\
0 & 0 & 1 & 2 & 1 & 2
\end{pmatrix} \]

Use block decompositions:

\[ \det(A) = \det\begin{pmatrix}1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \det\begin{pmatrix}3 & 7 \\ 0 & 0 \end{pmatrix} \cdot \det\begin{pmatrix}1 & 2 \\ 2 & 3 \end{pmatrix} \]

\[ = (1-4) \cdot (9 - 14) \cdot (4 - 3) \cdot (-3) \cdot (-2) \cdot 1 = 6 \]
Let $V$ be the three dimensional vector space of all symmetric, bilinear maps $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$. Let $A$ be the matrix \[
abla \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \].

Let $T : V \to V$ be the linear map which sends a symmetric bilinear map $(v, w) \mapsto s(v, w)$ to $(v, w) \mapsto s(Av, Aw)$. Find the trace of $T$. (Hint: Choose a basis for $V$ and compute the matrix of $V$ with respect to that basis and read off the trace).

Given any $s$, we can write $s$ as a linear combo of that basis. The coeffs are just what you get if you plug in those vectors, ie.

\[
\begin{align*}
s &= s \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot s_1 + s \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot s_2 + s \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot s_3
\end{align*}
\]

Let us compute $T(s_1)$:

\[
\begin{align*}
T(s_1) &= s_1 \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} \cdot s_1 + s_1 \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \cdot s_2 + s_1 \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \cdot s_3 \\
&= 1 + 2 \cdot 6 + 4 \cdot 10 \\
&= 48
\end{align*}
\]

\[
\begin{align*}
T(s_2) &= s_2 \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \cdot s_1 + s_2 \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \cdot s_2 + s_2 \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \cdot s_3 \\
&= 2 + 4 \cdot 6 + 4 \cdot 10 \\
&= 8 + 8 = 16
\end{align*}
\]

\[
\begin{align*}
T(s_3) &= s_3 \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \cdot s_1 + s_3 \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \cdot s_2 + s_3 \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \cdot s_3 \\
&= 4 + 12 + 16 \\
&= 24
\end{align*}
\]
So the matrix is

$$\begin{pmatrix}
1 & 6 & 9 \\
2 & 10 & 12 \\
4 & 16 & 16
\end{pmatrix}$$

whose trace is 27.
6. Let \( M_2(\mathbb{R}) \) be the vector space of \( 2 \times 2 \) matrices with real entries and the inner product \( \langle A, B \rangle = \text{tr}(A^*B) \) where \( \text{tr} \) denotes the trace of a matrix (sum of the diagonal entries). Let \( f : M_2(\mathbb{R}) \to M_2(\mathbb{R}) \quad A \mapsto A + A^* \)

Note that the elementary matrices
\[
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]
are an orthonormal basis (ONB).

1. Show that \( f \) is symmetric.

2. Compute \( f^2 \) and deduce that 0 and 2 are the only Eigenvalues of \( f \).

3. What is the rank of \( f \)?

4. Find an ONB of the range of \( f \).

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

\[
\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
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\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
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\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
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\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
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\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
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\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
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\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
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\[
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

\[
\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]
\[ q_3 = \frac{e_3}{\|e_3\|} = \frac{1}{\sqrt{\left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right)^2}} \cdot \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) = \frac{1}{\sqrt{2}} \cdot \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right). \]

Alternative: Look at the map \( \varphi: M_2(\mathbb{R}) \to M_2(\mathbb{R}) \) \( A \mapsto A^* \). Let us check that \( \varphi \) is self-adjoint, i.e., we have to show that for any \( A, B \in M_2(\mathbb{R}) \)

\[ <\varphi(A), B> = <A, \varphi(B)> \quad \text{for any } A, B \in M_2(\mathbb{R}) \]

\[ <\varphi(A), B> = tr\left( (A^*)^* B \right) = tr\left( A^* B \right) \quad \text{property of trace.} \]

\[ <A, \varphi(B)> = tr\left( A^* B^* \right) = tr\left( (BA)^* \right) = tr\left( (BA) \right) \quad \text{sum of diag. entries, order does not change} \]

\[ <A, \varphi(B)> = tr\left( A^* (B+B^*) \right) = tr\left( (A^* B + B^* A) \right) = tr\left( (A^* B) + (B^* A) \right) \]

\[ = tr\left( (B^* A^*) \right) = tr\left( (AB) \right) \]

\[ \Rightarrow \varphi \text{ is symmetric} \Rightarrow f = \varphi + id \text{ is symmetric.} \]

Another one: to show \( <\varphi(A), B> = <A, f(B)> \) for all \( A, B \).

\[ <\varphi(A), B> = <A + A^*, B> = tr\left( (A + A^*)^* B \right) = tr\left( A^* B \right) + tr\left( AB \right) \]

\[ <A, f(B)> = <A, B + B^*> = tr\left( A^* (B + B^*) \right) = tr\left( A^* B + A^* B^* \right) = tr\left( A^* B + B^* A \right) \]

\[ = tr\left( (B^* A^*) \right) = tr\left( (AB) \right) \]
7. Let \( V \) be a finite-dimensional complex inner product space. A linear map \( f : V \to V \) is called antisymmetric, if \( f^* = -f \). Show that if \( f \) is antisymmetric, \( V \) has an orthonormal basis consisting of Eigenvectors of \( f \). (Hint: Which property of a map of complex inner product spaces is equivalent to having an ONB of Eigenvectors?)
8. Let us look at the following matrix

\[ A = \begin{pmatrix} 4 & 1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}. \]

1. Compute the characteristic polynomial of \( A \) and factor it.

2. Find the Jordan Normal Form of \( A \).

3. Find a basis of \( \mathbb{R}^3 \) that gives that normal form.

\[ \text{det} \begin{pmatrix} 4-t & 1 \\ -1 & 2-t \\ 0 & 0 & 3-t \end{pmatrix} = (4-t)(2-t) + 1 = (3-t) = (t^2 - 6t + 9)(3-t) = -(t-3)^3. \]

\[ \tilde{A} = A - 3I = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 0 & 0 \end{pmatrix}, \quad \tilde{A}^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{A}^3 = 0 \]

\[ \dim(N(A^n)) \quad 1 \quad 2 \quad 3 \]

\[ \text{JNF} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} \]