1. Let $V$ be an $\mathbb{R}$-vector space and let $V^*$ be the vector space of all linear maps from $V$ to $\mathbb{R}$ (which is also denoted sometimes by $\text{Hom}(V, \mathbb{R})$). The vector space $V^*$ is also called the dual space of $V$. Let $B$ be a basis of $V$. For each basis vector $b \in B$, let $f_b$ denote the linear map $V \to \mathbb{R}$ that sends that basis vector to 1 and all other basis vectors from $B$ to 0. The list of those maps $\{f_b \mid b \in B\}$ is also called the dual basis of $B$.

1. Show that if $V$ is finite-dimensional, then the dual basis is a basis for the dual space.

2. Let $g : V \to W$ be a map of finite-dimensional vector spaces. Then $g$ induces a linear map (usually called $g^*$) between the dual spaces in the other direction sending an element $f : W \to \mathbb{R}$ of the dual space of $W$ to the element $f \circ g : V \to \mathbb{R}$ in the dualspace of $V$. Suppose that the matrix of $g$ with respect to some chosen bases is

$$
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{pmatrix}.
$$

Find the matrix of $g^*$ with respect to the dual bases.

3. Show that if $V$ is infinite-dimensional, then the dual basis is NOT a basis for the dual space (Find a linear map that cannot be written as a linear combination of the maps in the dual basis).

2. Given two vector spaces $V, W$. Let $\text{Hom}(V, W)$ denote the vector space of all linear maps from $V$ to $W$. Proof that the map $\Psi : V^* \times W \to \text{Hom}(V, W)$ sending a pair $f, w$ to the map $V \to W$ sending $v \in V$ to $f(v) \cdot w$ is bilinear.