1. [3 marks] Proof the rank-nullity theorem, i.e. if $f : V \to W$ is linear and $\dim(V) < \infty$, then
   \[ \dim(V) = \dim(N(f)) + \dim(R(f)). \]
   (Being able to choose bases as we like is great!)

2. [7 marks] The goal of this question is to show the uniqueness of the reduced row echelon form (proof also works for the reduced column echelon form). Given any linear map $f : \mathbb{R}^n \to V$, where $V$ is a finite-dimensional vector space. We have seen an algorithm that produces a basis of $V$ such that the matrix of $f$ with respect to this basis and the standard basis of $\mathbb{R}^n$ has reduced row echelon form.

   1. The first step is to show that the pivot elements are on the same positions.
      
      Let $W_k$ denote the subspace of $\mathbb{R}^n$ spanned by the first $k$ vectors of the standard basis ($W_0 = 0$). Now consider the images of these subspaces under $f$.
      
      Show that $\dim(f(W_{k+1})) > \dim(f(W_k))$ holds if and only if there is a pivot entry in the $k$-th column of the matrix of $f$ in reduced row echelon form.
      
      This gives an intrinsic characterisation of the position of the pivot elements, i.e. a characterisation that does not depend on the choice of basis of $V$.

   2. Suppose $A, B$ are two reduced row echelon forms for a map $f$. Recall that this means that they differ by left multiplication with a square matrix $M$, i.e. $MA = B$.
      
      By the previous part we know that the pivot elements occur in the same columns in $A$ and $B$. Show that $M \cdot e_k = e_k$ where $e_k$ is the $k$-th vector of the standard basis.

   3. Conclude that $MA = A$ and thus $A = B$. 