Recall: All the normal forms that we have learned so far:

\[ f: V \rightarrow V \text{ linear}, \quad b_1, \ldots, b_n \text{ & } c_1, \ldots, c_n \text{ two bases of } V. \]

\[ \Rightarrow \left[ \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right] \mapsto \left[ \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right] = \left[ \begin{array}{c} e_1 \\ \vdots \\ e_n \end{array} \right] = \left[ \begin{array}{c} y_1 \\ \vdots \\ y_n \end{array} \right] \]

Matrix of \( f \) w.r.t. this basis.

these two matrices are inverse to each other.

Notation: \( GL_n(F) \): invertible \( n \times n \) matrices over \( F \)
\( O_n(F) \): orthogonal \( n \times n \) matrices over \( F \) (makes sense for \( \mathbb{R}/\mathbb{C} \))
\( M_n(F) \): all \( n \times n \) matrices over \( F \).

1. \( \text{RREF: } f: V \rightarrow W \text{ given, Basis of } V \text{ given.} \)

Then there is a basis \( C \) of \( W \) s.t. \( \left[ f \right]_C^C \) has \( \text{RREF & RREF is unique} \)

Matrix version: \( A : m \times n \text{-matrix over } F \)

Then there is a \( B \in GL_m(F) \) s.t. \( BA \) has \( \text{RREF & RREF is unique} \)

(and for two matrices \( A, A' \) we can find a \( B \in GL_m(F) \) with \( BA = A' \) iff \( A \) and \( A' \) have the same \( \text{RREF's} \).)

2. \( \text{RCEF: } f: V \rightarrow W \text{ and Basis of } W \text{ given.} \)

Then there is a basis of \( V \) s.t. \( \left[ f \right]_C^C \) has \( \text{RCEF & uniquenesses} \)

Matrix version: \( A : m \times n \text{-matrix over } F \)

Then there is a \( B \in GL_n(F) \) s.t. \( AB \) has \( \text{RCEF} \) and for two matrices \( A, A' \) we can find \( B \in GL_n(F) \) with \( AB = A' \) iff \( A \) and \( A' \) have the same \( \text{RCEF's} \).
3) \( f : V \to W \) be given.

Then there are bases \( B \) of \( V \) and \( C \) of \( W \) such that
\[
[f]_B^C = \begin{pmatrix}
\phi & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

Matrix Version: \( A \): \( m \times n \) matrix over \( \mathbb{F} \). Then there is
\( B \in \text{GL}_m(\mathbb{F}) \) & \( C \in \text{GL}_n(\mathbb{F}) \) : \( BC = \begin{pmatrix}
\phi & 0 \\
0 & 0
\end{pmatrix}
\)

For given \( A \), \( A' \) we can find \( B \in \text{GL}_m(\mathbb{F}) \) & \( C \in \text{GL}_n(\mathbb{F}) \) with
\( BC = A' \) \( \Rightarrow \) Their normal forms are the same,
shorter invariant: \( \text{Rank}(A) \),
& all \( \text{Evalues} \) are real.

4) \( f : V \to V \) symmetric. Then there is an ONB of \( \text{Eigen} \) vectors:

Matrix Version: Given a symmetric matrix \( A \) over \( \mathbb{F} \) (\( \mathbb{R} \) or \( \mathbb{C} \)). Then there is an \( \Theta \) \( B \in \text{O}_n(\mathbb{F}) \) with
\( BAB^{-1} = \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2 \\
0 & \lambda_3 \\
0 & \lambda_4 \\
\end{pmatrix}
\)

For two given symmetric matrices \( A \), we can find \( B \in \text{O}_n(\mathbb{F}) \) with
\( BAB^{-1} = A' \), if they have the same \( \text{Eigen} \) values.
shorter invariant: \( \text{characteristic Polynomial of } A \).

Uniqueness up to rearranging the \( \text{Evalues} \).
(5) \( f : V \to V \) symmetric (\( V \) complex finite-dimensional inner product space).

Then there is an ONB of eigenvectors and all \( \lambda \) values have abs value 1. 

Uniqueness: up to reordering the eigenvalues.

Matrix Version: Given \( A \in O_n(C) \), then there is a \( B \in O_n(C) \):

\[
B A B^{-1} = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \\ \vdots & \ddots & \ddots \\ 0 & \cdots & 0 & z_n \end{pmatrix}
\]

with \( z_i \in \mathbb{C}, \ |z_i| = 1 \).

Shorter invariant: For two given matrices \( A, A' \in O_n(C) \), we can find a \( B \in O_n(C) \) with \( B A B^{-1} = A' \) iff they have the same eigenvalues.

Shorter invariant: characteristic polynomial of \( A \).

(6) \( f : V \to V \) orthogonal (\( V \) real finite-dim. inner product space).

Then there is an ONB \( \mathcal{B} \) of \( V \) s.t. \( [f]_{\mathcal{B}} = \begin{pmatrix} A_1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & A_n \end{pmatrix} \) where \( A_i \) is

- either a \( 1 \times 1 \) block with entry \( \pm 1 \)
- or a \( 2 \times 2 \) block of the form \( \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \) for some \( t \in [0, \pi] \).

Uniqueness: up to reordering the blocks.

Matrix Version: For \( A, A' \in O_n(R) \), there is a \( B \in O_n(R) \) with

\[
B A B^{-1} = \begin{pmatrix} A_1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & A_n \end{pmatrix}
\]

with \( A_i \) as above.

For given \( A, A' \in O_n(R) \), we can find a \( B \in O_n(R) \) with \( B A B^{-1} = A' \) iff they have the same \( \lambda \) values.

Shorter invariant: characteristic polynomial.
7. \( f : V \to V \) linear / C
(to be continued...) \( \Rightarrow \) Jordan Normal Form.