Solving linear equations

Given a system of linear equations, e.g.,

\[
\begin{align*}
&x_1 + 2x_2 + 3x_3 = 10 \\
&x_1 + x_2 + x_3 = 6 \\
&x_1 - x_2 + x_3 = 2
\end{align*}
\]

We can write it in the form

\[
\begin{pmatrix}
1 & 2 & 3 \\
1 & 1 & 1 \\
1 & -1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{pmatrix} =
\begin{pmatrix}
10 \\
6 \\
2 \\
\end{pmatrix}
\]

We want to get rid of those entries.

We can multiply this equation by matrices from the left (i.e.,
we can perform row operations). We want a lot of zeros:

\[
\begin{pmatrix}
1 & 2 & 3 \\
0 & -1 & 2 \\
0 & -3 & -2 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{pmatrix} =
\begin{pmatrix}
10 \\
-4 \\
-8 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 3 & -1 \\
0 & 1 & 2 \\
0 & 0 & 4 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{pmatrix} =
\begin{pmatrix}
2 \\
4 \\
4 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
10 & 0 & 0 \\
6 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{pmatrix} =
\begin{pmatrix}
3 \\
2 \\
1 \\
\end{pmatrix}
\]

So we have found the solution!

This matrix notation saves a lot of time. We do not have to write down a stupid amount of "+ - signs or "\(x_i\)’s."

But we still have "(\(x_i\))" in every line. Some people go further and just write

\[
\begin{pmatrix}
1 & 2 & 3 & | & 10 \\
1 & 1 & 1 & | & 6 \\
1 & -1 & 1 & | & 2 \\
\end{pmatrix}
\]

for the first equation (and so on...)
Elimination

So let me describe what we just did in a flowchart.
At any point in time we look at the whole matrix and but we also focus on a submatrix.

Start with submatrix = whole matrix

Is the submatrix empty (no entries)? (NOT: All entries = 0)

YES -> DONE

NO

Is there a nonzero entry in the first column of my submatrix?

YES

Swap that row to the top of our submatrix

Divide that row by that entry (to make it = 1)

Add multiples of that row to all other rows to make in the WHOLE (not) all other entries in the current column = equal to 0

NO

Leave out the first column of the submatrix and start again
Remark: When we want to solve \((A16)\) we start with submatrix \(A\).

So after we apply this process to a matrix how will the result look like? We cannot get rid of these entries, since they are only 0 below them.

We also might have entries here that we cannot get rid of.

might start with some columns 0.

So what does this tell us?

Let us have a look at one example:

Find \(x_1, x_2, x_5\) s.t.

\[
\begin{align*}
x_2 + 2x_3 + 3x_4 &= 5 \\
x_5 &= 2 \\
0 &= b_3
\end{align*}
\]

\(\Rightarrow 0 \text{ if } b_3 \neq 0 \Rightarrow \text{No solutions}\)

Otherwise \(x_5 = 2\), \(x_3\) and \(x_4\) are arbitrary, but you have to choose \(x_2 = 5 - 2x_3 - 3x_4\) accordingly.

Remark: The underlined entries are called the pivot elements of the matrix.
Said differently, 
\[
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_5
\end{pmatrix} = \begin{pmatrix}
  1 \\
  0 \\
  0 \\
  0 \\
  0
\end{pmatrix}
\] is one solution and any other solution can be obtained by adding multiples of
\[
\begin{pmatrix}
  -2 \\
  0 \\
  0 \\
  0 \\
  0
\end{pmatrix}
\] and
\[
\begin{pmatrix}
  -3 \\
  0 \\
  0 \\
  0 \\
  0
\end{pmatrix}
\] to it.

Exercise: Solve
\[
\begin{pmatrix}
  1 & 2 & 4 & 6 \\
  1 & 3 & 7 & 9 \\
  2 & 5 & 3 & 13
\end{pmatrix}
\] 
\[
\begin{pmatrix}
  1 & 2 & 4 & 6 \\
  1 & 3 & 7 & 9 \\
  2 & 5 & 3 & 13
\end{pmatrix}
\] 
No swap, No division
\[
\begin{pmatrix}
  1 & 2 & 4 & 6 \\
  1 & 3 & 7 & 9 \\
  2 & 5 & 3 & 13
\end{pmatrix}
\] 
Adding multiples of 1st row
\[
\begin{pmatrix}
  1 & 0 & 2 & 6 \\
  0 & 1 & 3 & 3 \\
  0 & 1 & 5 & 1
\end{pmatrix}
\] 
Submatrix is empty \Rightarrow DONE

\[
\begin{pmatrix}
  1 & 2 & 4 & 6 \\
  1 & 3 & 7 & 9 \\
  2 & 5 & 3 & 13
\end{pmatrix}
\] 
Submatrix is empty \Rightarrow DONE

\[
\begin{pmatrix}
  1 & 0 & 2 & 6 \\
  0 & 1 & 3 & 3 \\
  0 & 1 & 5 & 1
\end{pmatrix}
\] 
\[
\begin{pmatrix}
  1 & 2 & 4 & 6 \\
  1 & 3 & 7 & 9 \\
  2 & 5 & 3 & 13
\end{pmatrix}
\] 
Submatrix is empty \Rightarrow DONE

One solution is
\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5
\end{pmatrix} = \begin{pmatrix}
  -2 \\
  0 \\
  0 \\
  1 \\
  0
\end{pmatrix}
\] and any other solution is obtained by adding multiples of
\[
\begin{pmatrix}
  -2 \\
  -5
\end{pmatrix}
\] to it.