Vector spaces

Let $F$ be a fixed field, e.g., the real numbers.

A lot of mathematical structures carry naturally an addition and a way of rescaling elements. For example, the set of all functions $\mathbb{R} \to \mathbb{R}$ or the set of all continuous functions $f: \mathbb{R} \to \mathbb{R}$, or the set of all $n$-tuples of real numbers.

These addition and multiplication should of course be compatible with the field structure on $\mathbb{R}$ in the following sense.

Def.: An $F$-vector space is a set $V$ together with two operations $+: V \times V \to V$ (Addition)

$\cdot: F \times V \to V$ such that the following properties hold:

(US1) [Commutativity of $+$] $x + y = y + x$ for all $x, y \in V$

(US2) [Associativity of $+$] $(x + y) + z = x + (y + z)$ for all $x, y, z \in V$

(US3) [Existence of O] There is an elt $0 \in V$ with $0 + x = x$ for all $x \in V$

(US4) [Existence of additive inverses] For each $x \in V$ there is $y \in V$ with $x + y = 0$

(US5) For each $x \in V$: $1 \cdot x = x$

(US6) For $a, b \in F$, $x \in V$: $(a \cdot b) \cdot x = a \cdot (b \cdot x)$

(US7) Right Distributive law of $\cdot$: For $a \in F$, $x, y \in V$: $a \cdot (x + y) = a \cdot x + a \cdot y$

(US8) Left Distributive law of $\cdot$: For $a, b \in F$, $x \in V$: $(a + b) \cdot x = a \cdot x + b \cdot x$. 
Elements of $F$ are called **scalars** and elements of $V$ are called **vectors**.

**Examples:**
1) The set $F^n$ of $n$-tuples of elements of $F$ forms a **vector space**. Addition is defined via:
\[
(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)
\]
(We will usually write tuples in this way.)

Scalar multiplication is defined by $f \cdot (x_1, \ldots, x_n) = (f \cdot x_1, \ldots, f \cdot x_n)$

2) The set of continuous functions $\mathbb{R} \to \mathbb{R}$ is a **vector space**.
3) The set of differentiable functions $\mathbb{R} \to \mathbb{R}$ is a **vector space**.
4) The set of all functions $\mathbb{R} \to \mathbb{R}$ is a **vector space**.
5) The set of (all/bounded/...) sequences of real numbers is a $\mathbb{R}$-**vector space**.

The following definitions are really important: \{v_1, \ldots, v_n\}

1) A **linear combination** of a set of vectors is a vector of the form $\sum_{i=1}^{n} \lambda_i v_i$ with $\lambda_i \in F$.

2) A set $S \subseteq V$ is called a **generating system** if every vector can be written as a linear combination of elements of $S$, i.e.,
\[ V = \left\{ \sum_{i=1}^{n} \lambda_i v_i \mid \lambda_i \in F, \right\} = \langle S \rangle 
\]
3) A subset $S$ is called \textit{linearly independent}, if there is no nontrivial way to write the zero vector as a linear combination (i.e. if the following implication holds):

$$\lambda_1 v_1 + \ldots + \lambda_n v_n = 0 \implies \lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$$

4) A \textbf{basis} of $V$ is a linearly independent generating set.

\textbf{Remark: 1)} Any element in $V$ can be written in a unique way as a linear combination of elements of the basis.

After choosing a basis we can identify a vector $v = a_1 b_1 + \ldots + a_n b_n$ with the list of numbers $(a_1, \ldots, a_n)$.

\textbf{2)} A linear combination is always a finite sum.

3) The empty sum is $0$. So $\langle \emptyset \rangle = 0$

\textbf{Example:} The standard basis of $\mathbb{F}^n$: $(1, 0, \ldots, 0)$, $(0, 1, \ldots, 0)$, $(0, \ldots, 1)$

(eg. for $\mathbb{R}^3$: $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$)

\textbf{Remark:} All problems in this lecture course can be solved by looking at a different basis.
Example:

\[ B = \{ (2), (1) \frac{2}{3} \} \text{ is a basis of } \mathbb{R}^2. \]

We will now express the vector \( \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} \) as a linear combination of the other basis:

\[ \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} = \frac{3}{2} \cdot (2) + \frac{3}{2} \cdot (1) \]

In some sense, changing the basis is just looking at the situation from a different angle.