SOLUTIONS TO HOMEWORK ASSIGNMENT #1

1. Sketch the curve \( r = 1 + \cos \theta, 0 \leq \theta \leq 2\pi \), and find the area it encloses.

![Figure 1: The curve \( r = 1 + \cos \theta, 0 \leq \theta \leq 2\pi \)](image)

The area is given by

\[
A = \frac{1}{2} \int_{\theta=0}^{2\pi} (1 + \cos \theta)^2 \, d\theta = \frac{1}{2} \int_{\theta=0}^{2\pi} (1 + 2 \cos \theta + \cos^2 \theta) \, d\theta \\
= \frac{1}{2} (2\pi + 0 + \pi) = \frac{3\pi}{2}
\]

2. Find the dot product \( \vec{a} \cdot \vec{b} \) in the following cases:

(a) \( \vec{a} = \langle 1, 0, -2 \rangle, \vec{b} = \langle 2, 0, 1 \rangle \). Are these vectors orthogonal?

(b) \( \vec{a} = \langle x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1 \rangle, \vec{b} = \langle x_1, x_2, x_3 \rangle \), where the \( x_i, y_i \) are any real numbers. Are these vectors orthogonal?

(c) \( \vec{a} \) is a unit vector having the same direction as \( \vec{i} + \vec{j} \) and \( \vec{b} \) is a vector of magnitude 2 in the direction of \( \vec{i} + \vec{j} - \vec{k} \).

Solution:

(a) \( \vec{a} \cdot \vec{b} = 2 - 2 = 0 \). These vectors are orthogonal.

(b) \( \vec{a} \cdot \vec{b} = (x_2y_3 - x_3y_2)x_1 + (x_3y_1 - x_1y_3)x_2 + (x_1y_2 - x_2y_1)x_3 = 0 \). These vectors are orthogonal.

(c) \( \vec{a} = \frac{1}{\sqrt{2}}(\vec{i} + \vec{j}) \) and \( \vec{b} = \frac{2}{\sqrt{3}}(\vec{i} + \vec{j} - \vec{k}) \) and therefore \( \vec{a} \cdot \vec{b} = \frac{2\sqrt{2}}{\sqrt{3}} \).
3. Use cross products to find the following areas:
   (a) the area of the triangle through the points \( P = (1, 1, 0), Q = (1, 0, 1), R = (0, 1, 1) \).
   (b) the area of the parallelogram spanned by the vectors \( \vec{u} = \langle 1, 2, 0 \rangle, \vec{v} = \langle a, b, c \rangle \).
   (c) the areas of all 4 faces of the tetrahedron whose vertices are \((0, 0, 0), (a, 0, 0), (0, b, 0)\) and \((0, 0, c)\), where \(a, b, c\) are positive numbers.

Solution:
(a) Let \( \vec{a} = Q - P = -\vec{j} + \vec{k}, \vec{b} = R - P = -\vec{i} + \vec{k}. \) Then the area of the triangle is
\[
\frac{1}{2} |\vec{a} \times \vec{b}| = \frac{1}{2} |(-\vec{j} + \vec{k}) \times (-\vec{i} + \vec{k})| = \frac{1}{2} |\vec{k} - \vec{i} - \vec{j}| = \frac{\sqrt{3}}{2}.
\]
(b) \( \vec{u} \times \vec{v} = (\vec{i} + 2\vec{j}) \times (a\vec{i} + b\vec{j} + c\vec{k}) = 2c\vec{i} - c\vec{j} + (b - 2a)\vec{k}. \) Thus the area is
\[
\sqrt{5c^2 + (b - 2a)^2} = \sqrt{4a^2 - 4ab + b^2 + 5c^2}.
\]
(c) The areas of the faces in the co-ordinate planes (i.e. the \(x, y\) plane, the \(y, z\) plane and the \(z, x\) plane) are \( \frac{ab}{2}, \frac{bc}{2}, \frac{ca}{2} \) respectively. To find the area of the sloping face we compute the cross product of the vectors \( \vec{a} = -a\vec{i} + c\vec{k}, \vec{v} = -a\vec{i} + b\vec{j} \):
\[
\vec{u} \times \vec{v} = (-a\vec{i} + c\vec{k}) \times (-a\vec{i} + b\vec{j}) = -bc\vec{i} - ca\vec{j} - ab\vec{k}.
\]
Thus the area of the sloping face is \( \frac{1}{2} \sqrt{(bc)^2 + (ca)^2 + (ab)^2} \).

4. Suppose \( \vec{a} \) is a vector in 3-space. Show that \( \left( \frac{\vec{a} \cdot \vec{i}}{|\vec{a}|} \right)^2 + \left( \frac{\vec{a} \cdot \vec{j}}{|\vec{a}|} \right)^2 + \left( \frac{\vec{a} \cdot \vec{k}}{|\vec{a}|} \right)^2 = 1 \).

Remark: The direction cosines of the vector \( \vec{a} \) are by definition
\[
\cos \alpha = \frac{\vec{a} \cdot \vec{i}}{|\vec{a}|}, \cos \beta = \frac{\vec{a} \cdot \vec{j}}{|\vec{a}|}, \cos \gamma = \frac{\vec{a} \cdot \vec{k}}{|\vec{a}|}.
\]
The angles \( \alpha, \beta, \gamma \) are the angles \( \vec{a} \) makes with the positive directions of the \(x, y, z\) axes respectively.

Solution:
Suppose \( \vec{a} = \langle a_1, a_2, a_3 \rangle \). Then
\[
\left( \frac{\vec{a} \cdot \vec{i}}{|\vec{a}|} \right)^2 + \left( \frac{\vec{a} \cdot \vec{j}}{|\vec{a}|} \right)^2 + \left( \frac{\vec{a} \cdot \vec{k}}{|\vec{a}|} \right)^2 = \left( \frac{a_1}{|\vec{a}|} \right)^2 + \left( \frac{a_2}{|\vec{a}|} \right)^2 + \left( \frac{a_3}{|\vec{a}|} \right)^2 = \frac{a_1^2 + a_2^2 + a_3^2}{|\vec{a}|^2} = 1.
\]

5. (a) Find all vectors of length 2 that make equal angles with the positive directions of the \(x, y, z\) axes respectively.
(b) Find all unit vectors \( \vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k} \) making respective angles of \( \pi/3, \pi/4 \) with the positive directions of the \( x, y \) axes.

(c) Find the angles of the triangle whose vertices are \((1, 0, 0), (0, 2, 0), (0, 0, 3)\).

(d) Find the angle(s) between a diagonal of a cube and one of its edges.

Solution:

(a) If \( \vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k} \) has length 2 and makes equal angles with respect to the positive directions of the 3 co-ordinate axes then

\[
\begin{align*}
v_1^2 + v_2^2 + v_3^2 &= 1, \\
v_1 &= \cos(\pi/3) = 1/2 \\
v_2 &= \cos(\pi/4) = 1/\sqrt{2}.
\end{align*}
\]

Therefore \( v_3^2 = 1 - 1/4 - 1/2 = 1/4 \), that is \( v_3 = \pm 1/2 \). Hence \( \vec{v} = (1/2, 1/\sqrt{2}, \pm 1/2) \).

(b) If \( \vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k} \) is a unit vector making angles \( \pi/3, \pi/4 \) with the positive directions of the \( x, y \) axes respectively then

\[
\begin{align*}
v_1^2 + v_2^2 + v_3^2 &= 4 \\
v_1 &= \lambda = \pm \frac{2}{\sqrt{3}} < 1, 1, 1 > = \pm \frac{2}{\sqrt{3}} (\vec{i} + \vec{j} + \vec{k}).
\end{align*}
\]

(c) Let \( P = (1, 0, 0), Q = (0, 2, 0), R = (0, 0, 3) \) and let \( \alpha, \beta, \gamma \) be the 3 angles at \( P, Q, R \) respectively. Then

\[
\begin{align*}
\cos \alpha &= \frac{\vec{PQ} \cdot \vec{PR}}{|\vec{PQ}| |\vec{PR}|} = \frac{1}{\sqrt{50}}, \\
\cos \beta &= \frac{\vec{QP} \cdot \vec{QR}}{|\vec{QP}| |\vec{QR}|} = \frac{4}{\sqrt{65}}, \\
\cos \gamma &= \frac{\vec{RP} \cdot \vec{RQ}}{|\vec{RP}| |\vec{RQ}|} = \frac{9}{\sqrt{130}}
\end{align*}
\]

Therefore \( \alpha \approx 1.428899272, \beta \approx 1.051650212, \gamma \approx 0.6610431690 \), all angles measured in radians. Note that \( \alpha + \beta + \gamma = \pi \).

(d) One of the diagonals of a (unit) cube is \( \vec{v} = \vec{i} + \vec{j} + \vec{k} \). The common angle \( \theta \) between \( \vec{v} \) and any of \( \vec{i}, \vec{j}, \vec{k} \) satisfies \( \cos \theta = \frac{\vec{v} \cdot \vec{i}}{\sqrt{3}} = \frac{1}{\sqrt{3}} \). Therefore \( \theta \approx 0.9553166180 \). The other possibility is the complementary angle, namely \( \pi - \theta \approx 2.186276036 \).

6. A straight river 400m wide flows due west at a constant speed of 3km/hr. If you can row your boat at 5km/hr in still water, what direction should you row in if you wish to go from a point \( A \) on the south shore to the point \( B \) directly opposite on the north shore? How long will the trip take?

Solution: We can take the velocity vector of the river to be \( \vec{v} = -3 \vec{i} \) and the “rowing” vector to be \( \vec{u} = 5(\cos \theta \vec{i} + \sin \theta \vec{j}) \), where \( \theta \) is the angle of inclination with respect to the east. We want \( \vec{v} + \vec{u} = (-3 + 5 \cos \theta) \vec{i} + (5 \sin \theta) \vec{j} \) to be a positive multiple of \( \vec{j} \). Therefore \( \cos \theta = 3/5 \) and \( \sin \theta = 4/5 \). That is \( \theta = \arccos(3/5) \approx 0.9272952180 \). With this choice of \( \theta \) our net velocity is \( 4 \vec{j} \). Therefore it will take \( \frac{1}{10} \) hr = 6 minutes to get to the opposite shore.
7. Find equations of the planes satisfying the following conditions:

(a) Passing through the point \((0, 2, -3)\) and normal to the vector \(4\vec{i} - \vec{j} - 2\vec{k}\).
(b) Passing through the point \((1, 2, 3)\) and parallel to the plane \(3x + y - 2z = 15\).
(c) Passing through the 3 points \((\lambda, 0, 0), (0, \mu, 0), (0, 0, \nu)\), where \(\lambda, \mu, \nu\) are non-zero real numbers.
(d) Passing through the point \((-2, 0, -1)\) and containing the line which is the intersection of the 2 planes \(2x + 3y - z = 0\) and \(x - 4y + 2z = -5\).

Solution:

(a) The equation is \(4(x - 0) - (y - 2) - 2(z + 3) = 0\), that is \(4x - y - 2z = 4\).
(b) The equation is \(3(x - 1) + (y - 2) - 2(z - 3) = 0\), that is \(3x + y - 2z = -1\).
(c) The equation is \(\frac{x}{\lambda} + \frac{y}{\mu} + \frac{z}{\nu} = 1\).
(d) The equation will have the form \(\mu(2x + 3y - z) + \nu(x - 4y + 2z + 5) = 0\) for an appropriate choice of \(\mu, \nu\). Putting \(x = -2, y = 0, z = -1\) into this equation gives \(-3\mu + \nu = 0\). Choosing \(\mu = 1, \nu = 3\) gives \(5x - 9y + 5z = -15\).

8. Let \(v_1 = (0, -1, 0), v_2 = (0, 1, 0), v_3, v_4\) be the 4 vertices of a regular tetrahedron. Suppose \(v_3 = (x, 0, 0)\) for some positive \(x\) and \(v_4\) has a positive \(z\) component. Find \(v_3\) and \(v_4\).

Solution: \(v_1, v_2, v_3\) must form an equilateral triangle in the \(x, y\) plane with each side having length 2. Therefore \(v_3 = (\sqrt{3}, 0, 0)\). The vertex \(v_4\) must lie over \(\frac{1}{3}(v_1 + v_2 + v_3) = \left(\frac{1}{\sqrt{3}}, 0, 0\right)\). Therefore \(v_4 = \left(\frac{1}{\sqrt{3}}, 0, z\right)\), where \(z\) is that positive number chosen such that the distance from \((0, 1, 0)\) to \(\left(\frac{1}{\sqrt{3}}, 0, z\right)\) is 2. Solving we get \(v_4 = \left(\frac{1}{\sqrt{3}}, 0, 2\sqrt{3}\right)\).