7.3 #2: Find the image of the boundary first. The boundary consists of two lines: R and R+it. Since ELT's maps circles/lines to circles/lines we need only check three points. \( f(i) = (2-i)i \) maps \( 0 \rightarrow \infty, 1 \rightarrow 1-i, \infty \rightarrow 1 \). So the image of \( R \) is a line through \( 1 \) and \( 1-i \). \( f(i) \) maps \( i \rightarrow 0 \), \( i + i \rightarrow \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i \), \( \infty \rightarrow 1 \), so the image is a circle centred at \( \frac{1}{2} \) with radius \( \frac{1}{2} \). Checking orientations, we can draw the image.

7.3 #3: (a) This is a translation. The image is \( |W-(2-2i)| = 1 \).

(b) This is a scaling. \( |z-2| = \left| \frac{3i}{2} - 6i \right| = \left| \frac{1}{3} \right| |W-6i| \).

So \( |z-2| = 1 \) \( \implies \) \( |W-6i| = 3 \).

(c) The image is a circle or line. Check 3 points: \( i \rightarrow \infty \), \( 2+i \rightarrow \frac{i(1-i)}{2} = \frac{1+i}{2} \), \( 3 \rightarrow \frac{1}{2} \), so the image is a vertical line through \( \frac{1}{2} \).

(d) \( i \rightarrow \frac{3}{2}, 2+i \rightarrow \frac{-2+i}{-1+i} = \frac{(-2+i)(-1-i)}{2} = \frac{2-i+1}{2} = \frac{3+i}{2}, 3 \rightarrow \infty \).

So the image is a vertical line through \( \frac{3}{2} \).
7.3 #6 Let \( z_1, z_2, z_3 \) be fixed points of a F.L.T. \( f(z) \). We know there is a unique F.L.T. \( g(z) \) mapping \( z_1 \rightarrow 0 \), \( z_2 \rightarrow 1 \) and \( z_3 \rightarrow \infty \).

Prove: Let \( g(z) = \frac{az+b}{cz+d} \) then \( g(z_1) = 0 \Rightarrow a z_1 + b = 0 \) and \( g(z_3) = \infty \Rightarrow c z_3 + d = 0 \), so \( g(z) = \frac{a z - a z_1}{c z - c z_3} = \left( \frac{a}{c} \right) \left( \frac{z - z_1}{z - z_3} \right) \). Now \( g(z_2) = 1 \) implies \( \left( \frac{a}{c} \right) = \frac{z_2 - z_1}{z_2 - z_3} \) so \( g(z) \) is determined. \( g \) of \( g^{-1}(z) \) is both.

Apply \( g^{-1}(z) \) to both sides to conclude \( f(z) = z \).

7.3 #9 Under the map \( w = z/(z-1) \) \( 0 \rightarrow 0 \), \( \infty \rightarrow 1 \) so the image of the boundary rays are arcs of circle through \( 0 \) and \( 1 \). Note \( 1+i \rightarrow 1-i \) so the image looks like this.

How can we find the centre of the circle? The brute force method would be to solve \( |a-0|^2 = |a-1|^2 = |a-i-1|^2 \) for \( a \). But we could also note the by conformality, the line tangent to the circle at \( 0 \) makes an angle of \( -\pi/4 \) with the \( x \) axis, so the diameter makes an angle of \( -\pi/4 \). So the midpoint is \( \frac{1-i}{2} \).
and the radius is \( \frac{11-11}{2} = \frac{1}{12} \), so the circle is \( 1 - \frac{1}{12}^{1} = \frac{1}{12} \).

The upper circle will be \( 1 - \frac{1}{2}^{1} = \frac{1}{12} \).

7.3 #11 Use the map \( f(z) = \frac{2\pi i z}{z - 2} \). Then \( z \to \infty, 0 \to 0, -2 \to 2\pi i \).

Since the circle only intersected at 2, the images will be parallel lines through 0 and \( 2\pi i \). Since \( f(-1) = \frac{\pi}{2} \) the image will lie between the lines. Thus \( \exp(z) \) will map this region to a strip.

7.4 #14 Let \( f(z) = (z, z_1, z_2, z_3) \) then \( f \circ T^{-1} \) maps

\( T(z_2) \to 0, T(z_3) \to 1, T(z_4) \to \infty \). Thus \( f(T^{-1}(z)) = (z, T(z_2), T(z_3), T(z_4)) \).

Now set \( z = T(z_2) \).

7.4 #8 If \( S \) and \( T \) both map \( z_1 \to \infty, z_2 \to z_3, z_3 \to z_1 \), then \( S^{-1} \circ T \) fixes \( z_1, z_2, z_3 \). Thus \( S^{-1} \circ T(z) = z \) (from previous problem), so \( T(z) = S(z) \).

7.4 #9 Since \( 1, -1 \) are symmetric with the imaginary axis, \( f(1) = \infty \) and \( f(-1) = 0 \) are symmetric with the unit circle. Thus \( f(-1) = 0 \).
1. \( f_{(a\ b)} \left( f_{(a'\ b')} \left( z \right) \right) = f_{(a'\ b')} \left( \frac{a'z + b'}{c'z + d'} \right) \)

\[ = \frac{a \left( a'z + b' \right) + b \left( c'z + d' \right)}{c \left( a'z + b' \right) + d \left( c'z + d' \right)} = \frac{(aa' + bc')z + (ab' + bd')}{(ca' + db')z + (cb' + dd')} \]

\[ = f_{(a\ b)} \left( (a'\ b') \left( z \right) \right) \]

2. \( f_{(a\ b)} \left( (z) \right) = \frac{a \cdot z + b}{c \cdot z + d} = \frac{az + b}{cz + d} = f_{(ab)} \left( z \right) \)

3. Since \( f_{A^{-1}} \circ f_{A} \left( z \right) = f_{AA} \left( z \right) = f_{I} \left( z \right) = z \),

\( f_{A^{-1}} = (f_{A})^{-1} \). Since \( (a\ b)^{-1} = \frac{1}{ad-bc} \left( \begin{array}{c} d \ -b \\ -c \ a \end{array} \right) \),

\( f_{(a\ b)}^{-1} = f_{\frac{1}{ad-bc}} \left( \begin{array}{c} -b \\ -c \ a \end{array} \right) = f_{(-c\ a)} \).